Multi-User Networks with Outdated Channel State Information

Dem Fachbereich 18
Elektrotechnik und Informationstechnik
der Technischen Universität Darmstadt
zur Erlangung der Würde eines
Doktor-Ingenieurs (Dr.-Ing.)
genehmigte Dissertation

von
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geboren am 14. Marz 1987 in Leningrad, UdSSR

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Tag der Einreichung:	26. Juni 2018
Tag der mündlichen Prüfung:	21. Januar 2019

D 17
Darmstädter Dissertation

Buzuverov, Alexey: Multi-User Networks with Outdated Channel State Information
Darmstadt, Technische Universität Darmstadt
Jahr der Veröffentlichung der Dissertation auf TUprints: 2019
URN: urn:nbn:de:tuda-tuprints-85716
Tag der mündlichen Prüfung: 21.01.2019

Acknowledgements

I would like to sincerely thank my supervisor Prof. Anja Klein, for letting me in into the scientific world and guiding me in my Ph.D. journey throughout four and a half years. I gratefully acknowledge the support provided after my scholarship has expired and all the feedback I got for my work.

I would like to thank Prof. Marc Pfetsch for agreeing to be my co-supervisor within the Graduate School of Computational Engineering. I would like to thank the Graduate School of Computational Engineering at Technische Universität Darmstadt for providing me a three-year scholarship and the coverage of the travel expenses.

I would like to thank Dr. Bruno Clerckx for hosting me in Imperial College London for three months. The fruitful discussions have helped me to progress in a new for me topic. I would like to thank Dr. Clerckx for agreeing to be the co-referee of this thesis.

My sincere thanks to Dr. Hussein Al-Shatri for supervising me during my first three years of the study. Our collaboration helped me to increase my writing standards a lot.

I would like to thank Dr. Hamdi Joudeh for supervising me during my three-month stay in Imperial College London and all the feedback I got for my work later on.

I would like to thank my colleagues in the Communication Engineering Lab for all the discussions we had on the countless group meetings and all the informal activity we were involved together. My big thanks to my room-mates Andrea and Tobias for having fun together and providing me with a pleasant working atmosphere.

Last but not least, I would like to thank my family in Russia and all my friends for supporting me on my way to completing this thesis. Without your help, this would be difficult.

Kurzfassung

Die Verbreitung allgegenwärtiger mobiler Breitbandkommunikation hat unseren Alltag und unsere Gesellschaft signifikant verändert. Der Einsatz mehrerer Antennen an Sendern und Empfängern, auch bekannt als Multiple-Input Multiple-Output (MIMO) Technologie, ist einer der Schlüssel für Fortschritte in der mobilen Kommunikation. Akkurate und aktuelle Kanalinformation auf der Senderseite (Channel State Information at the Transmitter, CSIT) ist dabei eine notwendige Voraussetzung, um Multiplexing-Gewinne, welche in der Literatur auch als Freiheitsgrade (Degrees of Freedom, DoF) bezeichnet werden, zu erzielen. Jedoch kann die Bereitstellung stets aktueller CSIT sehr aufwändig in Bezug auf die benötigten Ressourcen werden. Im Falle vollständig veralteter CSIT kann keine Kanal-Zeit-Korrelation ausgenutzt werden. Dennoch kann sogar vollständig veraltete CSIT sehr nützlich sein, um mehr DoF zu erzielen, als ohne jegliche CSIT möglich ist. Die Schlüsselidee ist dabei eine Mehrphasen-Übertragung, wobei in jeder Phase der mitgehörte Interferenzterm aus den vorherigen Übertagungsphasen erneut übertragen wird. Einerseits liefern diese Terme den Sendern neue Informationen über die gewünschten Symbole, andererseits können diese Terme auf Empfängerseite entfernt werden, wo sie zuvor mitgehört wurden. Dadurch kann die Menge an anfallender Interferenz in jeder der aufeinanderfolgenden Phasen reduziert werden, wobei in der letzten Phase eine interferenzfreie Übertragung erreicht wird. In dieser Arbeit entwerfen wir neue Übertragungsschemata, um für eine Vielzahl von Kommunikationsnetzwerken mit teilweise oder vollständig veralteter CSIT mehr DoF zu erzielen.

Zunächst wird ein Netzwerk mit zwei Sendern und zwei Empfängern betrachtet, in dem jeder Sender eine Nachricht an jeden Empfänger senden möchte. Ein solches Netzwerk wird in der Literatur auch als X-channel (XC) bezeichnet. Wir betrachten einen MIMO-Fall, in welchem die Sender über M_1 bzw. M_2 Antennen und die Empfänger über N_1 bzw. N_2 Antennen verfügen. Im XC empfängt jeder Empfänger, ausgehend von den beiden verschiedenen Sendern, eine Überlagerung zweier Interferenzsignale. Daher kann die Interferenz in dieser Form nicht durch veraltete CSIT rekonstruiert werden. Durch den Einsatz von Redundanzübertragung (RT), kann jeder Sender dazu gebracht werden, nur einen Bruchteil des Signalraums jedes Empfängers zu überspannen. Desweiteren kann jeder Empfänger das Signal von einem Störer durch den Einsatz von Partial Interference Nulling (PIN) subtrahieren, wobei das verbleibende Interferenzsignal durch den Einsatz von veralteter CSIT am Sender rekonstruiert werden kann. Für den Fall, dass $\min\{M_1, M_2\} > \min\{N_1, N_2\}$ gilt, kann eine effizientere mehrteilige Übertragung, auch bekannt als Interference Sensing and Redundancy

Transmission (IS-RT), durchgeführt werden, bei der die Interferenz, welche im ersten Teil empfangen wird, die Interferenz aus dem zweiten Teil beinhaltet. In dieser Arbeit führen wir Dekodierbarkeitsanalysen des State-of-the-art Übertragungsschemas für den MIMO XC mit veralteter CSIT durch, welche auf IS-RT-PIN beruht. Unsere Analysen zeigen, dass trotz der Tatsache, dass die Empfänger eine ausreichende Anzahl von Linearkombinationen erhalten, die übertragenen Informationssymbole aufgrund von linearer Abhängigkeit der Linearkombinationen nicht immer dekodierbar sind. Um das identifierte Dekodierbarkeitsproblem zu lösen, schlagen wir ein neues Übertragungsschema vor, bei welchem die Parameter sorgsam gewählt werden, um die erzielbaren DoF zu maximieren und gleichzeitig die lineare Unabhängigkeit sicherzustellen. Das vorgeschlagene Übertragungsschema erzielt eine größere Anzahl an DoF als das State-of-the-art Übertragungsschema, bei welchem die Anzahl an übertragenen Informationssymbolen auf die Anzahl an dekodiebaren Informationssymbolen reduziert wird.

Desweiteren wird ein Netzwerk mit drei Sendern und drei Empfängern betrachtet, in dem jeder Sender eine Nachricht an seinen jeweiligen Partner-Empfänger senden möchte. Ein solches Netzwerk wird in der Literatur als Drei-Nutzer Interference Channel (IC) bezeichnet. Wir betrachten einen symmetrischen MIMO-Fall, in dem jeder Sender M Antennen und jeder Empfänger N Antennen hat. Für den Drei-Nutzer MIMO IC mit veralteter CSIT werden zwei neue Übertragungsschemata für $M < N$ und $M > N$ vorgeschlagen, welche höhere DoF erzielen als aus der Literatur bekannte Verfahren. Das erste Übertragungsschema für $M < N$ beruht auf RT-PIN, wobei berücksichtigt wird, dass für $M < N$ die Redundanz natürlicherweise vom Kanal eingebracht wird. Das vorgestellte Verfahren hat eine Drei-Phasen-Struktur, wobei in jeder Phase die Menge an eingebrachter Redundanz entsprechend $\frac{M}{N}$ angepasst wird. Das zweite Übertragungsschema für $M > N$ beruht auf IS-RT-PIN. Wie bereits für den MIMO XC mit veralteter CSIT und den Verfahren basierend auf IS-RT-PIN identifiert, kann es aufgrund von linearen Abhängigkeiten zu einem Verlust an Dekodierbarkeit kommen. Das Übertragungsschema aus der Literatur nutzt in Phase 1 eine zweiteilige IS-RT, wobei die Anzahl an verwendeten Sendeantennen für ausreichend große $\frac{M}{N}$ begrenzt wird, um einen Verlust an Dekodierbarkeit zu verhindern. In diesem Fall werden die zusätzlichen Sendeantennen nicht ausgenutzt. Unser vorgeschlagenes Übertragungsschema nutzt stattdessen eine neue dreiteilige IS-RT in Phase 1, in welcher die IS und RT Teile verschiedener Sender eine unterschiedliche Länge haben. Eine solche Übertragung erlaubt die Reduzierung der Anzahl der linear abhängigen Linearkombinationen, wobei die Anzahl an genutzten Sendeantennen nur an einem einzigen Sender beschränkt ist. Die Parameter des vorgeschlagenen Übertragungsschemas werden sorgsam gewählt, um die erzielbaren DoF zu maximieren und gleichzeitig die lineare Unabhängigkeit sicherzustellen. Dabei wird eine größere

Anzahl an DoF erzielt als in der Literatur. Zusätzlich zu den zwei vorgeschlagenen Verfahren wird eine obere Grenze der linearen DoF präsentiert, welche sich als sehr eng für $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ und $2 \leq \frac{M}{N} < 3$ darstellt.

Außerdem wird der 2-Antennen 3-Nutzer Multiple-Input Single-Output (MISO) Broadcast Channel (BC) mit alternierender CSIT betrachtet, in welchem das CSIT für jeden Nutzer entweder perfekt (P) oder veraltet (D) sein kann, was in insgesamt 8 mögliche CSIT Zustände $I_1 I_2 I_3, I_i \in \{\mathrm{P}, \mathrm{D}\}, i \in \{1, 2, 3\}$, resultiert. Für dieses Szenario erhalten wir neue Ergebnisse über die DoF Charakterisierung. Das erste Ergebnis charakterisiert die DoF-Region für den Fall, in dem die CSIT Zustände die folgenden 5 Werte annehmen: PPP, PPD, PDP, PDD und DDD. Das zweite Ergebnis charakterisiert die DoF für den Fall, in dem die CSIT Zustände alle möglichen Werte annehmen können, die gemeinsamen CSIT Zustandswahrscheinlichkeiten jedoch bestimmte Verhältnisse erfüllen müssen. Um die optimalen DoF zu erzielen, wird eine gemeinsame Codierung über alle verfügbaren CSIT Zustände vorgeschlagen, was Gewinne im Vergleich zur individuellen Codierung über jeden einzelnen CSIT Zustand liefert. Um unsere Ergebnisse zu erhalten, schlagen wir zuerst vier neue konstituierende Kodierungsschemata (CSs) vor, welche eine gemeinsame Codierung der CSIT Zustands-Tupel (PPP,PDD), (PDD, DDD), (PDD,DPD,DDD) und (PDD,DPD,DDP) ausführen. Nach einer sorgfältigen Zuordnung der neu vorgeschlagenen und der in der Literatur existierenden CSs zu den verfügbaren CSIT Zuständen werden die optimalen DoF erzielt.

Abstract

The spread of ubiquitous high-speed mobile communication has changed our daily life and society significantly. Using multiple antennas at transmitters and receivers, known as multiple-input and multiple-output (MIMO) technology, is one of the key developments which allowed new advances in mobile communication. Accurate and up-to-date channel state information at the transmitter (CSIT) is a necessary requirement for achieving the multiplexing gains, referred to in the literature also as degrees of freedom (DoF). Maintaining up-to-date CSIT however may become exhausting in terms of the number of resources. In case the CSIT is completely outdated, no channel time correlation can be exploited. Nevertheless, even completely outdated CSIT can be very useful for achieving DoF greater than that with completely absent CSIT. The key idea is to apply a multi-phase transmission, where in each phase, the interference terms overheard in the previous phases are retransmitted. On one hand, such terms provide the transmitters with new information about the desired symbols. On the other hand, such terms can be cancelled at the receivers which previously overheard them. In such a way, the amount of the produced interference in each consecutive phase is reduced, where in the last phase, an interference-free transmission is achieved. In this thesis, we design new transmission schemes to achieve more DoF in a variety of communication networks with completely outdated or simply delayed CSIT.

Firstly, a network with two transmitters and two receivers is considered, where each transmitter desires to deliver a message to each receiver. Such network is referred to in the literature as the X-channel (XC). We consider a MIMO setting, in which the transmitters have M_1 and M_2 antennas and the receivers have N_1 and N_2 antennas. In the XC, each receiver receives a superposition of two interference signals originating from different transmitters, hence the interference in its plain form cannot be reconstructed using delayed CSIT. By applying redundancy transmission (RT), each transmitter can be forced to span only a fraction of the signal space of each receiver. Then, by applying partial interference nulling (PIN), each receiver can subtract the signal of one of the interferers, where the remaining interference signal can be reconstructed at the transmitter using delayed CSIT. In case $\min\{M_1, M_2\} > \min\{N_1, N_2\}$, a more effective multi-part transmission, known as interference sensing and redundancy transmission (IS-RT), can be performed, where the interference overheard in the first part comprises the redundancy transmitted in the second part. In this thesis, we perform decodability analysis of the state-of-the-art transmission scheme for the MIMO XC with delayed CSIT which relies on IS-RT-PIN. Our analysis shows, that despite the fact that the receivers obtain a sufficient number of linear combinations, the transmitted information symbols are not always decodable, which is due to a linear dependence of the

linear combinations. To address the identified decodability problem, a novel transmission scheme is proposed, where the parameters of the scheme are carefully selected to maximize the number of the transmitted information symbols while ensuring linear independence. The proposed transmission scheme achieves a number of DoF greater than that of the state-of-the-art transmission scheme in which the number of the transmitted information symbols is reduced to the number of the decodable ones.

Secondly, a network with three transmitters and three receivers is considered, where each transmitter wants to deliver a message to its corresponding partner receiver. Such network is referred to in the literature as the three-user interference channel (IC). We consider a symmetric MIMO setting, in which each transmitter has M antennas and each receiver has N antennas. For the three-user MIMO IC with delayed CSIT, two novel transmission schemes for $M < N$ and $M > N$ are proposed which achieve DoF greater than that in the literature. The first transmission scheme proposed for $M < N$ relies on RT-PIN, where we take into account the fact that for $M < N$, the redundancy is naturally introduced by the channel. The proposed transmission scheme has a three-phase structure, where in each phase the amount of the introduced redundancy is adjusted according to $\frac{M}{N}$. The second transmission scheme proposed for $M > N$ relies on IS-RT-PIN. As already identified for the MIMO XC with delayed CSIT, for the transmission schemes relying on IS-RT-PIN, a loss of decodability due to linear dependencies of linear combinations may occur. The transmission scheme existing in the literature uses in phase 1 a two-part IS-RT, where to avoid loss of decodability, the number of used transmit antennas is limited for sufficiently large $\frac{M}{N}$. In such case, the additional transmit antennas are not exploited. Our proposed transmission scheme, instead, uses in phase 1 a novel three-part IS-RT, in which the IS and RT parts of different transmitters have different durations. Such transmission allows to reduce the number of linearly dependent linear combinations, while the number of used transmit antennas is limited only at a single transmitter. The parameters of the proposed transmission scheme are carefully selected to maximize the number of the transmitted information symbols while ensuring linear independence. A number of DoF greater than that in the literature is achieved. In addition to the two proposed transmission schemes, an upper bound on the linear DoF is proposed, which turns out to be tight for $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ and $2 \leq \frac{M}{N} < 3$.

Thirdly, the 2-antenna 3-user multiple-input single-output (MISO) broadcast channel (BC) with alternating CSIT is considered, in which the CSIT for each user can be either perfect (P) or delayed (D), resulting thus in total in 8 possible CSIT states $I_1 I_2 I_3$, $I_i \in \{\mathrm{P}, \mathrm{D}\}$, $i \in \{1, 2, 3\}$. For this scenario, we obtain two new results on the DoF characterization. The first result characterizes the DoF region for the case where the CSIT states can take the following 5 values: PPP, PPD, PDP, PDD and

DDD. The second result characterizes the DoF for the case where the CSIT states can take all possible values, however the joint CSIT state probabilities are restricted to fulfil certain relationships. To achieve the optimal DoF, joint encoding over the available CSIT states is proposed, which provides DoF gains as compared to encoding over each CSIT state independently. To obtain our results, we first propose four novel constituent encoding schemes (CSs), which perform joint encoding of the CSIT state tuples (PPP, PDD), (PDD, DDD), (PDD, DPD, DDD) and (PDD, DPD, DDP). Then, after a careful assignment of the newly proposed and existing in the literature CSs to the available CSIT states, the optimal DoF are achieved.

Contents

1 Introduction **1**
1.1 Motivation 1
1.2 State-of-the-Art 5
1.2.1 Delayed CSIT 6
1.2.1.1 Broadcast Channel 6
1.2.1.2 Interference Channel and X-Channel 6
1.2.2 Alternating CSIT 9
1.3 Open Issues 11
1.4 Contributions and Overview 12

2 Achievable DoF of the MIMO X-channel with Delayed CSIT **15**
2.1 Introduction 15
2.2 System Model 16
2.3 Achievability Scheme 18
2.3.1 Overview 18
2.3.2 Transmission Blocks 19
2.3.3 Phase 1 20
2.4 Decodability Analysis 22
2.5 Proposed Transmission Scheme 25
2.6 Achieved Number of DoF 26
2.7 Conclusion 30

3 Achievable DoF of the 3-User Symmetric MIMO Interference Channel with Delayed CSIT **33**
3.1 Introduction 33
3.2 System Model 34
3.3 Main Results 37
3.4 Proof of Theorem 1 40
3.4.1 Transmission Blocks 40
3.4.2 Case of $M < N$ 41
3.4.2.1 Phase 1 42
3.4.2.2 Phase 2 44
3.4.2.3 Phase 3 46
3.4.2.4 Numbers of Transmission Blocks and Achieved DoF . . . 48
3.4.3 Case of $M > N$ 48
3.4.3.1 IS-RT Approach 48
3.4.3.2 Phase 1 50

3.4.3.3 Phase 2 54
3.4.3.4 Phase 3 57
3.4.3.5 Numbers of Transmission Blocks and Achieved DoF . . 58
3.5 Conclusion 58

4 The DoF of the 2-Antenna 3-User MISO Broadcast Channel with Alternating CSIT 61
4.1 Introduction 61
4.2 System Model 62
4.3 Main Results 64
4.4 Constituent Encoding Schemes 66
4.4.1 Schemes Achieving 2 DoF 66
4.4.2 Schemes Achieving $\frac{5}{3}$ DoF 68
4.4.2.1 $S_1^{5/3}$: phase 1 70
4.4.2.2 $S_1^{5/3}$: phase 2 72
4.4.3 Schemes Achieving $\frac{3}{2}$ DoF 74
4.4.4 Scheme Achieving $\frac{12}{7}$ DoF 74
4.4.5 Scheme Achieving $\frac{33}{19}$ DoF 76
4.5 Achievability for Theorem 3 79
4.5.1 Achievability of the Optimal DoF Corner Points 79
4.5.2 Achievability of the Non-Optimal DoF Corner Points 80
4.5.2.1 Summary of the DoF Corner Points 80
4.5.2.2 Achievability 87
4.6 Proof of Theorem 4 88
4.7 Conclusion 92

5 Conclusions and Outlook 95
5.1 Conclusions 95
5.2 Outlook 96

Appendix 99

A Appendix 99
A.1 Proof of Theorem 2 99
A.2 Proof of Lemma 2 101
A.3 Proof of Lemma 3 104

List of Acronyms 107

List of Symbols 109

Bibliography **115**

Own Publications **121**

Supervised Student Theses **123**

Lebenslauf **127**

Chapter 1

Introduction

1.1 Motivation

Access to high-speed mobile communication became the standard of our daily life. With all the advances in mobile communication technology, the demands are ever growing with the number of the users in the network increasing [CIS16]. The interference has been found to be the main limiting factor impacting the performance of communication systems [CO13]. Since decades, the interference has been handled by assignment of orthogonal resources to independent transmissions, which for the individual user leads however to the loss of the throughput proportional to the number of simultaneously active users [CO13]. Using multiple antennas at transmitters and receivers, known as multiple-input multiple-output (MIMO) technology, addresses this issue by separating independent transmissions rather in spatial domain [CO13]. This permits active users in the network to reuse all spectral resources simultaneously, resulting thus in a multiplicative increase in the overall throughput. The gain in the throughput as compared to single antenna point-to-point transmission is often measured by a metric called degrees of freedom (DoF), which is the ratio of the sum-rate over the logarithm of the transmit signal power P in the limit of large P. For the M-antenna K-user multiple-input single-output (MISO) broadcast channel (BC), in [CS03] the number of DoF was found to be $\min\{M, K\}$, which was achieved by zero-forcing (ZF) encoding. Since then, MIMO has been adapted to a variety of communication networks to achieve a greater number of DoF [WSS06, JF07, MAMK08, JS08, CJ09, GJ10, WGJ14].

The multiplexing gains achieved using MIMO come, however, at a cost of the requirement for channel state information at the transmitter (CSIT). In practice, CSI is obtained at the receiver side via pilot training. Unfortunately, there is no natural way to obtain CSI at the transmitter side. To overcome this, feedback links can be arranged to supply the transmitter with CSI obtained at the receivers. To maintain the full DoF, it is sufficient to have feedback links with a rate which scales linearly with the signal-to-noise ratio (SNR) in dB [CJS07, Jin06, CJKR10]. Having time-varying channels, however, puts further challenges on the CSIT acquisition, since by the time the CSIT has been obtained, the channel state could have already changed. In case when the feedback delay is comparable to the coherence time, the CSIT becomes completely outdated. In this worst-case scenario, the transmitter has no way to determine how his signals cause interference at other receivers, hence interference cannot be managed using ZF [HJSV12]. At first glance, the completely outdated CSIT, or, as referred to in the literature, delayed CSIT is of no use as it is completely irrelevant to the actual channel state.

Surprisingly, [MAT12] has shown that delayed CSIT can still be very useful to achieve the DoF greater than that in case of completely absent CSIT. For the K-user MISO BC with delayed CSIT, [MAT12] has shown the DoF to be $\frac{K}{1+\frac{1}{2}+\cdots+\frac{1}{K}}$, which is greater than that 1 DoF in case of no CSIT [VV12a]. To achieve the DoF gains, delayed CSIT was used as side-information, which allowed the transmitter to reconstruct the previously overheard interferences. The reconstructed interference terms were then used to create the future transmissions which provided new information to the intended receivers while aligning the received signals with the past overheard interferences at the unintended receivers.

The following motivating example shown in Figure 1.1 demonstrates the benefits of using delayed CSIT for achieving more DoF in the 2-user MISO BC, referred to as MAT scheme in the following. In the example, a transmission spanning 3 channel uses $t = 1, 2, 3$ is considered, over which 2 symbols $u_1^{[1]}$ and $u_1^{[2]}$ are delivered to Rx_1 and 2 symbols $u_2^{[1]}$ and $u_2^{[2]}$ are delivered to Rx_2. Optimal $\frac{4}{3}$ DoF are achieved. The transmission is split into 2 phases: phase 1 comprising channel uses $t = 1, 2$ and phase 2 comprising channel use $t = 3$. In phase 1, the symbols are transmitted without any CSIT: at $t = 1$, $u_1^{[1]}$ and $u_1^{[2]}$ are transmitted to Rx_1 and at $t = 2$, $u_2^{[1]}$ and $u_2^{[2]}$ are transmitted to Rx_2. At $t = 1$, Rx_1 receives useful linear combination $L_1^{[1]}(u_1^{[1]}, u_1^{[2]})$ and Rx_2 receives an interference term $L_2^{[1]}(u_1^{[1]}, u_1^{[2]})$ which is useful for Rx_1. At $t = 2$, the situation is symmetric, where Rx_2 receives a useful linear combination and Rx_1 receives an interference term which is useful for Rx_2. In phase 2, the delayed CSIT comes into play where the interference terms overheard in phase 1 are reconstructed at the transmitter. The reconstructed interference terms are combined into a signal of common interest for both receivers $u_{12} = L_2^{[1]}(u_1^{[1]}, u_1^{[2]}) + L_1^{[2]}(u_2^{[1]}, u_2^{[2]})$, referred to as order-2 symbol in the following. The delivery of u_{12} is performed at $t = 3$ by means of simple broadcasting. At the receiver side, the known interference is cancelled, where the receivers obtain remaining linear combinations necessary for decoding.

The success of [MAT12] motivates the use of delayed CSIT for achieving more DoF in types of networks other than the BC. For example, consider the following two networks depicted in Figures 1.2 and 1.3, referred to as the X-channel (XC) and the three-user interference channel (IC), respectively. In the XC, there are two transmitters and two receivers, where each of the transmitters wants to deliver independent messages to both receivers. In the three-user IC, there are three transmitters and three receivers, where

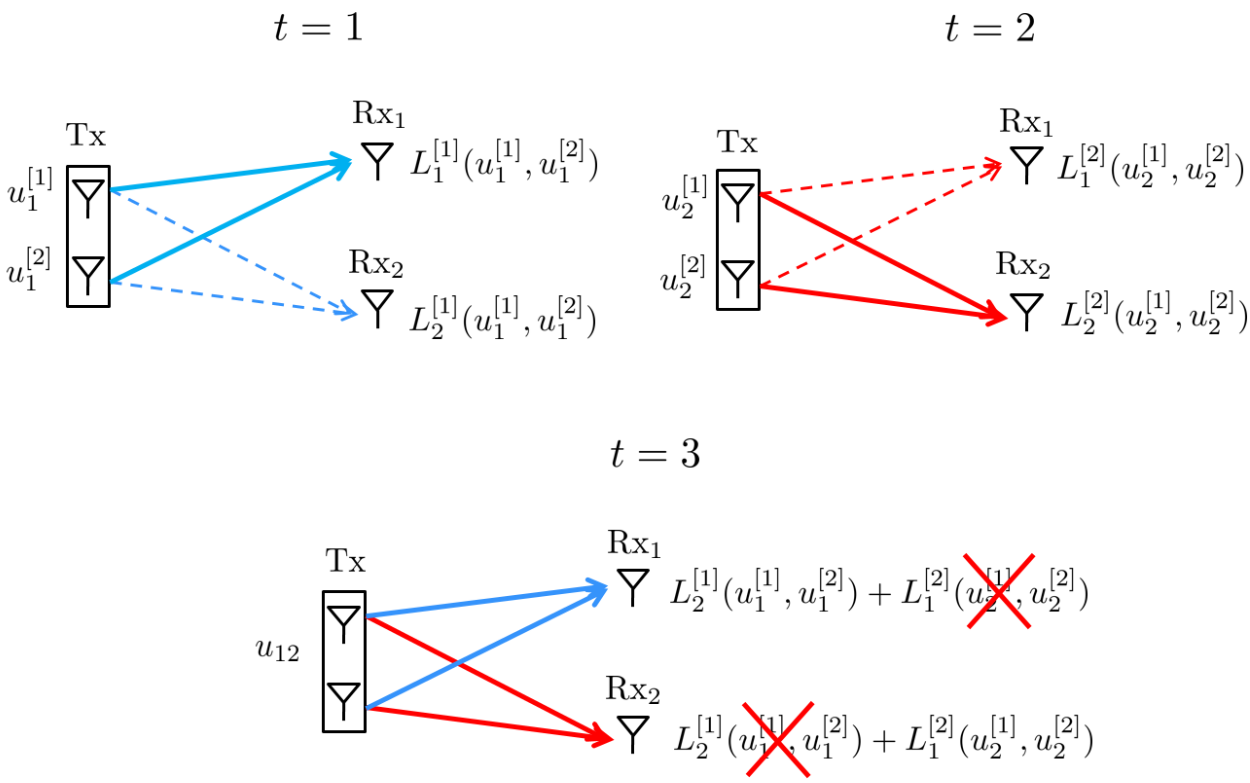

Figure 1.1: MAT scheme for $K = 2$

each transmitter wants to deliver a message to its corresponding partner receiver. In case of perfect CSIT, the number of DoF of the single-input single-output (SISO) XC is $\frac{4}{3}$ [MAMK08,JS08] and the number of DoF of the three-user SISO IC is $\frac{3}{2}$ [CJ09], which are greater than 1 DoF in case of no CSIT for both networks [VV12a] and achieved using interference alignment. Hence, it is interesting whether a number of DoF greater that 1 can be obtained with delayed CSIT as well. Unfortunately, it turns out that the MAT scheme is not directly applicable to the XC and the IC. This stems from the fact that in XC and IC, the interference at each receiver is due to multiple transmitters which do not share the transmitted information symbols. As a consequence, the interference signal cannot be reconstructed at the transmitter when it contains other transmitters' signals. To overcome these limitations, [MJS12] proposed an approach referred to as redundancy transmission (RT) and partial interference nulling (PIN), which is illustrated in Figure 1.4. The key idea is to transmit the information symbols along with some redundancy which forces the transmitters to occupy only a fraction of the receive signal space of the unintended receiver. By projecting the received signal onto a corresponding complementary subspace, the receiver is able to nullify one of the interference signals, where the remaining signal belongs to only a single transmitter and can be reconstructed using delayed CSIT. Using RT-PIN, $\frac{9}{8}$ DoF has been achieved in [MJS12].

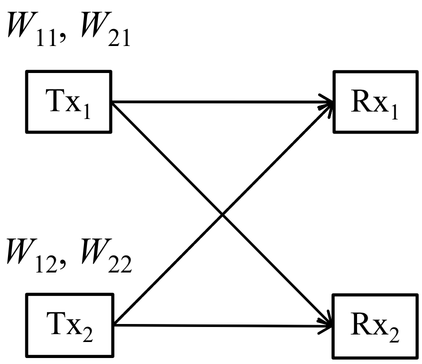

Figure 1.2: The X-channel. W_{ji} denotes the message to be delivered from Tx_i to Rx_i, $i, j \in \{1, 2\}$.

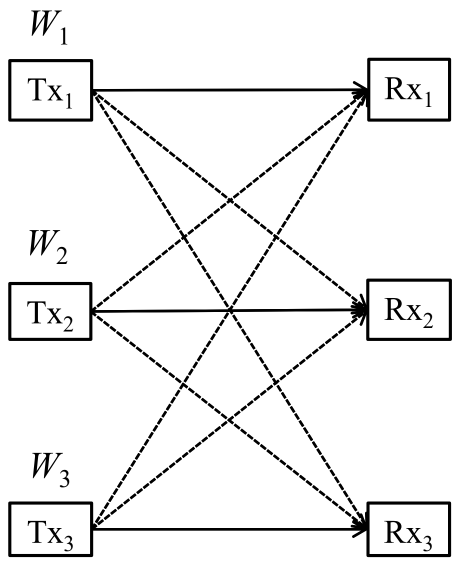

Figure 1.3: The three-user interference channel. W_i denotes the message to be delivered from Tx_i to Rx_i, $i = 1, 2, 3$.

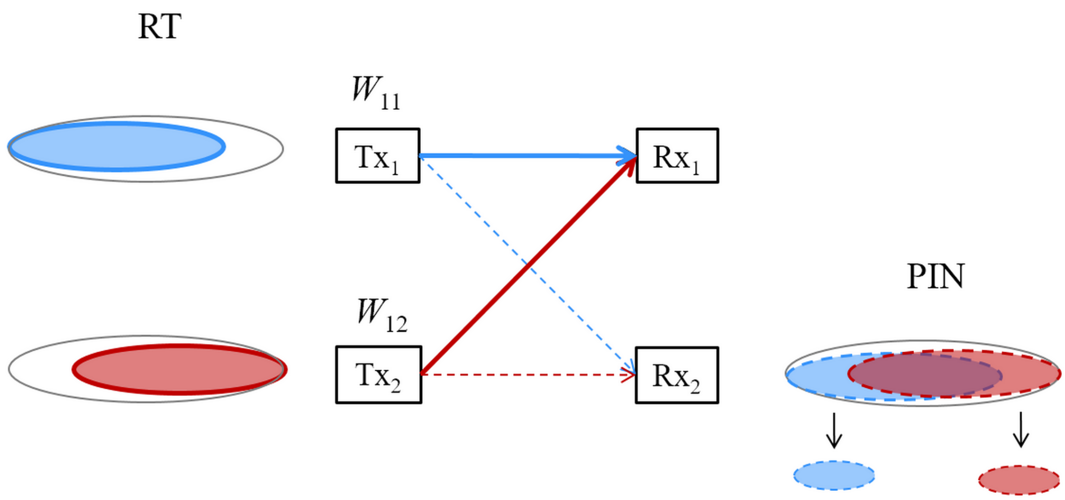

Figure 1.4: Redundancy Transmission and Partial Interference Nulling in the SISO XC. Here, both transmitters apply RT such that their signals span only a fraction of the receive signal space of Rx_2. Rx_2 applies PIN to obtain the interference terms containing the signals of only Tx_1 or Tx_2.

The work of [MJS12] considered the setting where the transmitters and receivers have single antennas. Using multiple antennas is expected to provide DoF gains a compared to the single-antenna case. Although the DoF of the MIMO XC with delayed CSIT are fairly well studied, the DoF of the MIMO IC with delayed CSIT is yet an open question. The DoF achievability results for the SISO case can be readily extended to the multi-antenna case by restricting each transmitter and each receiver to use $\min\{M, N\}$ antennas. In this case, the number of transmit and receive dimensions, as well as the total DoF, scale by a factor of $\min\{M, N\}$. Such simplistic approach comes however at a price, since additional transmit and receive antennas are not exploited. Although there are works addressing this issue in the literature, exploiting additional transmit and receive antennas in the MIMO IC with delayed CSIT is yet an open question. Another question relates to the optimality of the DoF achieving schemes. The optimality of the DoF achieving scheme can only be stated if an upper bound is provided, matching the achieved DoF. In this case, the upper bound is deemed to be tight and the achievability scheme achieves the optimal DoF. For the DoF schemes existing in the literature, no matching upper bounds have been provided, hence, their optimality is still not known.

The case when receivers have different feedback links may lead to a broader CSIT setting, where the CSI provided by receivers is of different quality. Furthermore, due to the time-varying nature of the channel, the CSIT quality for each user may vary in time, leading to a setting known in the literature as alternating CSIT. One of the simplest instances of such setting is the case where the CSIT for each user can be either perfect (P) or delayed (D). For the 2-user MISO BC, this would result in total in 4 possible CSIT states: PP, PD, DP and DD. In such setting, it is interesting whether joint coding over different CSIT states may provide gains as compared to coding over each CSIT state independently. This question has been answered affirmatively in [TJSSP13], where the DoF of the 2-user MISO BC with alternating CSIT has been fully characterized. Moving beyond the 2-user case however, the DoF characterization of the MISO BC with alternating CSIT becomes challenging, where optimal DoF have been established only for particular CSIT configurations. To determine the DoF of the K-user MISO BC with alternating CSIT for arbitrary CSIT configurations is still an open question.

1.2 State-of-the-Art

In this section, an overview of the publications related to delayed and alternating CSIT is given.

1.2.1 Delayed CSIT

In this section, the overview of publications related to delayed CSIT is given.

1.2.1.1 Broadcast Channel

As we already mentioned in Section 1.1, the benefit of delayed CSIT for achieving more DoF has been demonstrated in the seminal work of [MAT12]. For achieving the optimal DoF for the K-user case, a generalized K-phase version of the transmission scheme depicted in Fig. 1.1 was proposed, in which in each phase, symbols of higher order have been generated. This allows to reduce the amount of the interference in each phase which results in a scaling of the DoF with the number K of users. For the outer bound, a genie-aided technique has been employed, where the signals of some receivers have been provided to other receivers. After applying such technique, a network known as the physically degraded BC was obtained, where feedback does not increase the capacity [Gam78]. Hence, the DoF of the physically degraded BC with no CSIT upper bound that of the original BC with delayed CSIT.

The work of [MAT12] considered the MISO BC. A more general setting of MIMO BC has been considered in [VV11], where the DoF for the 2-user case have been completely characterized. [AGK11] partially characterized the DoF for the 3-user MIMO BC. The precoder optimization for the MISO BC with delayed CSIT has been considered in [YG13] and [LR15]. Space-time block coding for the 2-user MISO BC with delayed CSIT has been studied in [CG15]. A constant-gap capacity approximation of the 2-user MISO BC with delayed CSIT has been established in [VMAA13]. The secure DoF (SDoF) of the 2-user MIMO BC with delayed CSIT has been characterized in [YKPS13]. The usefulness of the schemes for delayed CSIT in non-outdated CSIT scenarios has been shown in [XAJ12].

1.2.1.2 Interference Channel and X-Channel

In this section, the publications related to IC and XC with delayed CSIT are considered. The review of publications related to the SISO XC and IC will be followed by the review of the publications related to the MIMO XC and IC. Publications considering relevant settings of output feedback and delayed CSIT will be mentioned in the end.

SISO: As we previously mentioned in Section 1.1, exploiting delayed CSIT in the XC and IC is challenging. Based on the RT-PIN approach, [MJS12] has achieved $\frac{9}{8}$ DoF for the three-user SISO IC and $\frac{8}{7}$ for the SISO XC. For achieving the $\frac{9}{8}$ DoF in three-user SISO IC, a two phase transmission has been proposed. In phase 1, the information symbols are transmitted using RT, where the unintended receivers apply PIN. From the interference terms obtained using PIN, order-2 symbols are generated. The transmission of the generated order-2 symbols is performed in phase 2, where a subset of order-2 symbols was carefully selected for the retransmission to avoid additional interference in the order-2 symbols delivery.

The work of [GMK11] has improved upon [MJS12] by achieving $\frac{6}{5}$ DoF for the SISO XC. [AGK13] has considered the K-user SISO IC and the $2 \times K$ SISO XC for which a new number of DoF has been obtained. For the 3-user SISO IC, $\frac{36}{31}$ DoF have been achieved, improving thus upon the scheme in [MJS12]. Contrary to [MJS12], [AGK13] proposes to retransmit the generated order-2 symbols in two phases, referred to as phases 2 and 3, where delayed CSIT is exploited to mitigate the produced interference. For the initial order-2 symbol transmission in phase 2, pairs of transmitters are scheduled to transmit order-2 symbols useful for the same pair of receivers, where the remaining third receiver overhears a sum of two interference terms. Using the RT-PIN procedure, interference terms comprised of the signals of only a single transmitter are obtained at the unintended receiver in phase 2. The obtained terms are reconstructed using the delayed CSIT and are retransmitted in phase 3, where interference-free transmission is achieved.

Prior works have focused on achieving a higher number of DoF, but no optimality has been stated. One of the successful attempts to address this has been undertaken in [LAS14] by providing an upper bound matching $\frac{6}{5}$ DoF obtained in [GMK11]. For obtaining the upper bound, a restriction of linear coding strategies has been used, thus a characterization in terms of linear DoF (LDoF) has been established. The key idea to obtain the DoF upper bound was to upper bound the ratio of sizes of useful to interference signal spaces. Additionally, by using this bound, an upper bound of $\frac{9}{7}$ LDoF for the 3-user SISO IC has been obtained.

MIMO: The DoF of the 2-user MIMO IC have been completely characterized in [VV12b]. For $M < N$, [GAK12] has shown in the context of the symmetric MIMO XC, that additional receive antennas can be exploited by taking into account the fact that when $M < N$, the redundancy in the transmission is naturally introduced by the channel. By adjusting the amount of the additionally introduced redundancy according to the ratio $\frac{M}{N}$, [GAK12] obtained partial DoF characterization of the symmetric MIMO XC with delayed CSIT. The three-user symmetric MIMO IC with delayed CSIT has been considered in [TAV14], for which [TAV14] designed a transmission scheme based on the scheme [MJS12] which achieved $\frac{9}{8}$ DoF in the SISO case. [AGK13] however, achieved a higher number $\frac{36}{31}$ of DoF in the SISO case, hence a better transmission scheme can designed. Fully exploiting the additional receive antennas in the three-user MIMO IC with delayed CSIT is hence still an open question.

While relying on RT in the three-user symmetric MIMO IC, additional transmit antennas are of no use for $M > N$. To exploit additional transmit antennas, [TAV16] initially proposed to schedule only a single transmitter in phase 1, having thus similarity to the MAT scheme. The delivery of the overheard interference terms has been performed in phase 2. A more effective approach to exploit additional transmit antennas has been proposed in the context of MIMO XC by [KA17]. We refer to the approach proposed in [KA17] as interference sensing (IS) and redundancy transmission (RT). The key idea idea of IS-RT is as follows. In plain RT, the transmitted redundancy is chosen randomly, where the sizes of the signal spaces of individual transmitters are equal at the intended and unintended receivers. Having $M > N$ however, may allow to reduce the sizes of the signal spaces at the unintended receiver. Simultaneously, to allow for PIN, the total receive signal space has to be occupied only fractionally. To achieve this, a two part transmission was proposed. In the first part called IS, the transmitters transmit information without redundancy, where signal spaces at the intended and unintended receivers are equal. In the second part called RT, the redundancy is transmitted, which is linearly dependent on the past receptions in part 1 of the unintended receiver. After the transmission, PIN is applied, where the overall procedure is referred to as IS-RT-PIN. An example of IS-RT for the MIMO XC is illustrated in Figure 1.5.

The work of [HC16] proposed a novel transmission scheme for the K-user symmetric MIMO IC, where in phase 1, IS-RT-PIN has been applied. The remaining phases follow a simpler RT-PIN. For IS-RT-PIN, the authors in [HC16] identified a problem with decodability of the transmitted information symbols, which is due to a linear dependence of the received linear combinations. We note, that a similar problem has been independently identified by the author of this thesis in context of the MIMO XC in [BASK16]. We will elaborate on this more in Section 1.3. To avoid loss of decodability, [HC16] proposed to restrict the number of the used transmit antennas for sufficiently large $\frac{M}{N}$, thus additional transmit antennas were not fully exploited using IS-RT-PIN. To exploit additional transmit antennas for large $\frac{M}{N}$, [HC16] proposed to switch from IS-RT-PIN to a transmission in which only two transmitters are scheduled to transmit simultaneously. Such a transmission technique is however inferior to IS-RT-PIN for small $\frac{M}{N}$. Fully exploiting additional transmit antennas for the MIMO IC with delayed CSIT is hence still an open question.

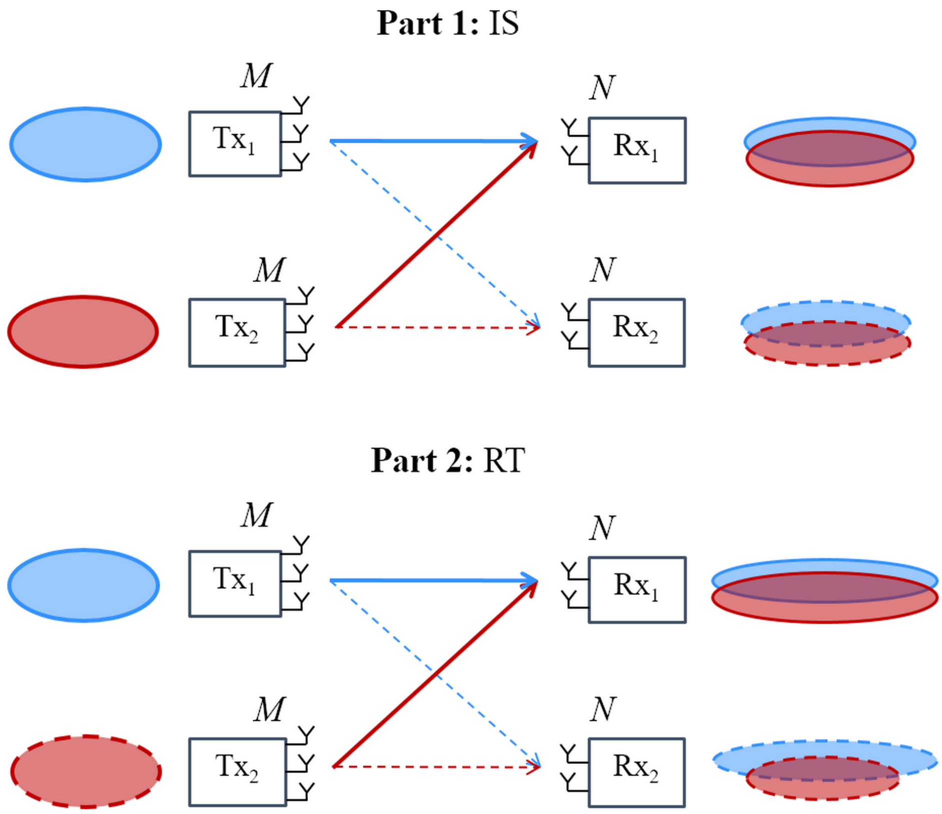

Figure 1.5: Interference sensing and redundancy transmission in the MIMO XC. Here, IS-RT is applied by Tx_2. As a result, the signal of Tx_2 occupies only a fraction of the signal space of Rx_2 and the size of the space at Rx_2 is reduced as compared to that at Rx_1.

Other relevant works: One of the modifications of the delayed CSIT setting is the one where in addition to delayed CSIT, the receivers provide their past received signals. Such setting is known in the literature as output feedback and delayed CSIT. Having output feedback may provide access to other transmitters' information symbols, thus allowing the reconstruction of interference signals containing symbols of other receivers. The DoF of the 2-user MIMO IC with output feedback and delayed CSIT has been completely characterized in [TMPS13] and [VV13]. The DoF of the symmetrical MIMO XC with output feedback and delayed CSIT has been established in [TMPS12]. Achievable DoF for the K-user SISO IC and $K \times K$ SISO XC with output feedback and delayed CSIT have been studied in [AGK15]. SDoF of the symmetric MIMO XC channel with output feedback and delayed CSIT have been characterized in [ZASV13].

1.2.2 Alternating CSIT

In this section, the overview of the publications related to alternating CSIT is given.

The works of [CS03], [MAT12] and [VV12a] considered the settings, where the CSIT for each user can be either perfect (P), delayed (D) or not available (N). The alternating CSIT setting, instead, assumes inhomogeneity of the CSIT for different users and allows the CSIT to alternate in time. [TJSSP13] fully characterized the DoF of the 2-user MISO BC with alternating CSIT. The K-user MISO BC with alternating CSIT has been studied in [TJSSP13], [RHC16] and [LH14], however optimal DoF were achieved only for particular CSIT configurations. The MISO BC where the CSIT is heterogeneous but fixed, referred to also as hybrid CSIT, has been studied in [ATS14] and [LTA16], where LDoF characterization for the 3-user case has been obtained. [LTA16] studied the K-user MISO BC with hybrid CSIT, where the optimal LDoF were established for the CSIT configuration in which only for a single receiver perfect CSIT is available. Achievable DoF of the 3-user IC and the $K \times 2$ XC with alternating CSIT have been studied in [LTH15].

The CSIT setting where the transmitter has a combination of imperfect instantaneous CSIT and perfect delayed CSIT generalizes the alternating CSIT setting in which the admissible CSIT states are either P or D. Such setting is referred to in the literature as delayed and imperfect CSIT. The DoF of the 2-user MISO BC with delayed and imperfect CSIT have been characterized in [YKGY13] for the homogeneous CSIT case. [CE13] established the DoF for the case where the CSIT quality is time-varying. The DoF of the 2-user MIMO BC with delayed and imperfect CSIT have been characterized in [YYGK14]. For the K-user MISO BC with delayed and imperfect CSIT, [dKYG13, dKGZE16] characterized the optimal DoF for the homogeneous CSIT case and [CYE13] considered the time-varying case. However, optimal DoF were achieved only for particular CSIT configurations. Precoder optimization for the MISO BC with delayed and imperfect CSIT has been considered in [WXWS13], [DC15].

Overloaded $M < K$ MISO BC: In this thesis we turn our attention to the overloaded $M < K$ MISO BC, where the CSIT for every user can be in either P or D state. Having $M < K$ makes the problem of DoF characterization more challenging, as for the MISO BC with delayed CSIT, [MAT12] achieved optimal DoF only in case $M = 2$, $K = 3$. Outer bounds for the MISO BC with alternating CSIT have been obtained in [TJSSP13] and [CYE13]. As for the achievability, [TJSSP13] achieved optimal $\min\{M, K\}$ DoF using ZF for the case where at least for M users, perfect CSIT is available. [CYE13] modified the setting in [TJSSP13] by allowing an alternation with the jointly delayed state in addition, where the optimal DoF have been achieved in case $M = 2$, $K = 3$. [LH14] considered the case of $M = K - 1$, where the CSIT alternates between jointly perfect and jointly delayed states. For this setting, [LH14] proposed a novel constituent encoding scheme (CS) which achieved optimal DoF in case $M = 2$, $K = 3$. [ATS14] obtained optimal DoF for the $M = 2$, $K = 3$ MISO BC with hybrid

PDD state. The case of $M = 2$, $K = 3$ is the only scenario for the $M < K$ MISO BC with alternating CSIT for which optimal DoF have been achieved in the literature. Despite the number of the results for the case $M = 2$, $K = 3$, the DoF characterization for arbitrary CSIT configurations is still an open question.

1.3 Open Issues

In this section, the open issues addressed in this thesis are given.

Firstly, the MIMO XC with delayed CSIT is considered. It is assumed that the transmitters have M_1 and M_2 antennas and the receivers have N_1 and N_2 antennas. The LDoF characterization of the MIMO XC has been obtained in [KA17]. For the case $\min\{M_1, M_2\} > \min\{N_1, N_2\}$ where IS-RT-PIN was applied, our simulations indicate that the information symbols transmitted using the scheme of [KA17] may not always be decodable, which is due to a linear dependence of the received linear combinations. Thereby, open issue 1 follows.

1. For the $\min\{M_1, M_2\} > \min\{N_1, N_2\}$ MIMO XC with delayed CSIT, how to exploit delayed CSIT to transmit more information symbols while ensuring their decodability?

Secondly, the symmetric three-user MIMO IC with delayed CSIT is considered, where each transmitter has M antennas and each receiver has N antennas. Existing schemes in the literature fail to fully exploit the additional receive antennas due to their reliance on either an inferior SISO transmission scheme structure or a receive antenna limitation. This results in open issue 2.

2. For the three-user $M < N$ MIMO IC with delayed CSIT, how to exploit additional receive antennas to achieve more DoF?

In case $M > N$, the RT-PIN approach is not effective, as the additional transmit antennas are not exploited. Instead, IS-RT-PIN has to be applied, which however, as we identified for the MIMO XC, is subject to a loss of decodability due to a linear dependence of the received linear combinations. Thereby, open issue 3 follows.

3. For the three-user $M > N$ MIMO IC with delayed CSIT, how to exploit additional transmit antennas to transmit more information symbols while ensuring their decodability?

Providing the achievability results does not yet result in the DoF characterization of the considered network. To establish optimal DoF, a matching upper bound is necessary. An assumption of linear coding strategies can weaken the DoF characterization results, yet it simplifies the upper bound derivation. Here, we come across open issue 4.

4. For the three-user MIMO IC with delayed CSIT, what are the optimal DoF assuming transmitters are restricted to use linear encoding strategies?

Thirdly, the 2-antenna 3-user MISO BC with alternating CSIT is considered, where the CSIT for each user can be either perfect (P) or delayed (D). This results in total in 8 possible CSIT states $I_1I_2I_3$, $I_i \in \{\mathrm{P}, \mathrm{D}\}$, $i \in \{1, 2, 3\}$. The existing works achieved optimal DoF only particular CSIT configurations, where a full DoF characterization is still missing. Thereby, the last open issue follows.

5. For the 2-antenna 3-user MISO BC with alternating CSIT, what are the optimal DoF?

1.4 Contributions and Overview

In this section, the contributions of the thesis and the overview of the thesis structure are given. For each open question from Section 1.3, the corresponding solution will be given.

In Chapter 2, the MIMO XC with delayed CSIT is considered. For the considered setting, we study the decodability of the information symbols transmitted using the transmission scheme [KA17] by evaluating linear independence of the received linear combinations. To achieve this, we provide a novel proof which is based on an upper bound on the rank of the effective channel matrix. When the proposed upper bound is less than the maximum rank, linear dependence can be stated. As a result, the loss of decodability for a region of antenna configurations is identified. To address the issue of linear dependence, we propose a novel transmission scheme. The parameters of the scheme are chosen to maximize the number of the transmitted information symbols while ensuring linear independence of the received linear combinations. The proposed transmission scheme achieves a number of DoF greater than that of [KA17] where the number of the transmitted information symbols is reduced to the number of the decodable ones. This solves open issue 1.

In Chapter 3, the three-user symmetric MIMO IC with delayed CSIT is considered. First, the case of $M < N$ is considered. For this scenario, a novel three-phase transmission based on RT-PIN is proposed. For the design of the scheme, the fact that for $M < N$, the redundancy is naturally introduced by the channel is taken into account, where the amount of the additionally introduced redundancy is adjusted in each phase depending on the ratio $\frac{M}{N}$. The proposed transmission scheme achieves a number of DoF greater than that in the literature. This solves open issue 2. For $M > N$, a novel transmission scheme based on IS-RT-PIN is proposed. As we already identified in Chapter 2, the transmission based on IS-RT-PIN is subject to a possible linear dependence. To overcome this, in phase 1 a novel three-part transmission is proposed, where IS and RT parts of different transmitters have different durations. Having asymmetry in the transmission reduces the number of linearly dependent linear combinations, where the number of transmit antennas is restricted only at a single transmitter. In phase 2, a two-part transmission based on IS-RT-PIN is applied. The parameters of the transmission in phases 1 and 2 are carefully chosen to maximize the number of the transmitted information symbols while ensuring linear independence of the received linear combinations. The proposed transmission scheme achieves a number of DoF greater than that in the literature. This solves the open issue 3. Additionally, an upper bound on the LDoF is provided. The bound is based on an upper bound on the ratio of useful to interference signal spaces. The tightness of the provided upper bound is demonstrated for $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ and $2 \leq \frac{M}{N} < 3$. This solves open issue 4.

In Chapter 4, the 2-antenna 3-user multiple-input single-output (MISO) broadcast channel (BC) with alternating CSIT is considered, in which the CSIT for each user can be either perfect (P) or delayed (D). For this scenario, we obtain two new results on the DoF characterization. The first result is a DoF region characterization for the CSIT setting where the admissible CSIT state set comprises the following 5 values: PPP, PPD, PDP, PDD and DDD. The second result characterizes the DoF for the case where the CSIT state set is not restricted, but the joint CSIT state probabilities are restricted to fulfil certain relationships. To obtain the optimal DoF, joint encoding over the available CSIT states is proposed. The achievability is facilitated through the introduction of four novel CSs, which perform joint encoding of the CSIT state tuples $(\mathrm{PPP}, \mathrm{PDD})$, $(\mathrm{PDD}, \mathrm{DDD})$, $(\mathrm{PDD}, \mathrm{DPD}, \mathrm{DDD})$ and $(\mathrm{PDD}, \mathrm{DPD}, \mathrm{DDP})$. Then, by an appropriate assignment of the CSs proposed and existing in the literature to the available CSIT states, the optimal DoF are achieved. This solves open issue 5.

Finally, the summary and conclusions of the thesis are given in Chapter 5.

Chapter 2

Achievable DoF of the MIMO X-channel with Delayed CSIT

2.1 Introduction

In this chapter, the XC channel with delayed CSIT is considered, where the transmitters have M_1 and M_2 antennas and the receivers have N_1 and N_2 antennas. As we already mentioned in Section 1.1, exploiting delayed CSIT to achieve more DoF in the XC is challenging, since the transmitters do not share the information symbols and hence each transmitter cannot reconstruct the interference using delayed CSIT when it contains the symbols of the other transmitter. To overcome this limitation, RT can be applied, which allows the unintended receiver to apply PIN and obtain terms containing the symbols of only a single transmitter. In case the transmitters have more antennas than the receivers, IS-RT can be applied, which allows to reduce the sizes of the signal spaces of individual transmitters at the unintended receiver and hence increase the achievable DoF.

In this chapter we consider the transmission scheme proposed in [KA17]. We restrict the antenna configurations to fulfil $\max\{N_1, N_2\} < M_1 + M_2$ and $\max\{M_1, M_2\} < N_1 + N_2$, which correspond to the case where both transmitters are active during the transmission and employ delayed CSIT for the transmission. Additionally, we focus our attention on the case $\min\{M_1, M_2\} > \min\{N_1, N_2\}$, where IS-RT is applied. For the transmission scheme proposed in [KA17], we perform an analysis of the decodability of the transmitted information symbols. In order to achieve this, we derive an upper bound on the rank of the effective channel matrix. We find that for particular antenna configurations, the obtained upper bound is smaller than the maximum rank which implies that the received linear combinations are linearly dependent. We address the problem of linear dependence by proposing a novel transmission scheme, in which the parameters of the transmission are chosen to maximize the number of the transmitted information symbols while ensuring linear independence. The proposed transmission scheme achieves a number of DoF greater than that of the transmission scheme [KA17] in which the number of the transmitted information symbols is reduced to the number of the decodable ones. Parts of the results of this chapter has been published by the author of this thesis in [BASK16].

The rest of the chapter is organized as follows. In Section 2.2, the system model is introduced. In Section 2.3, we describe the transmission scheme given in [KA17]. The decodability analysis is performed in Section 2.4 and the proposed transmission scheme is given in Section 2.5. The number of DoF achieved by the proposed transmission scheme is evaluated in Section 2.6. The conclusions of the chapter are given in Section 2.7.

2.2 System Model

In this section, the MIMO X-channel depicted in Figure 2.1 is considered. We assume that Tx_1 and Tx_2 have M_1 and M_2 antennas, respectively, and Rx_1 and Rx_2 have N_1 and N_2 antennas, respectively. Without loss of generality, we assume the transmitters and receivers to be ordered, such that $M_1 \geq M_2$ and $N_1 \geq N_2$ hold. With the assumption of the ordered transmitters and receivers, the constraints on the antenna configurations mentioned in Section 2.1 can be rewritten in a simplified form as $N_1 < M_1 + M_2$, $M_1 < N_1 + N_2$ and $M_2 > N_2$.

The signal received by Rx_j, $j \in \{1, 2\}$, at the t-th channel use is given by

$$\mathbf{y}_j(t) = \mathbf{H}_{j1}(t)\,\mathbf{x}_1(t) + \mathbf{H}_{j2}(t)\,\mathbf{x}_2(t) + \mathbf{z}_j(t) \in \mathbb{C}^{N_j \times 1}, \tag{2.1}$$

where $\mathbf{x}_i(t) \in \mathbb{C}^{M_i \times 1}$ is the signal transmitted by Tx_i, $i \in \{1, 2\}$, $\mathbf{H}_{ji}(t) \in \mathbb{C}^{N_j \times M_i}$ is the channel matrix between Tx_i and Rx_j, and $\mathbf{z}_j(t) \sim \mathcal{CN}\left(0, \mathbf{I}_{N_j}\right)$ is the additive white noise at Rx_j drawn from a complex Gaussian distribution with zero mean and identity covariance matrix. The channel coefficients are drawn from continuous distributions and are i.i.d. across different transmit and receive antennas and different channel uses. The signal transmitted by Tx_i is subject to the transmit power constraint $\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}\left\{\mathbf{x}_i^{\mathrm{H}}(t)\,\mathbf{x}_i(t)\right\} \leq P$, where n is the communication duration.

Let us denote the set of channel matrices up to the t-th channel use as

$$\mathcal{H}^t = \left\{\mathbf{H}_{ji}(\tau) \,|\, i, j \in \{1, 2\},\ \tau = 1, \ldots, t\right\}. \tag{2.2}$$

In the delayed CSIT setting, the following CSI knowledge is available at the transmitters and receivers:

- at the t-th channel use, $\mathcal{H}^t$ is known at every receiver;
- at the t-th channel use, $\mathcal{H}^{t-1}$ is known at every transmitter.

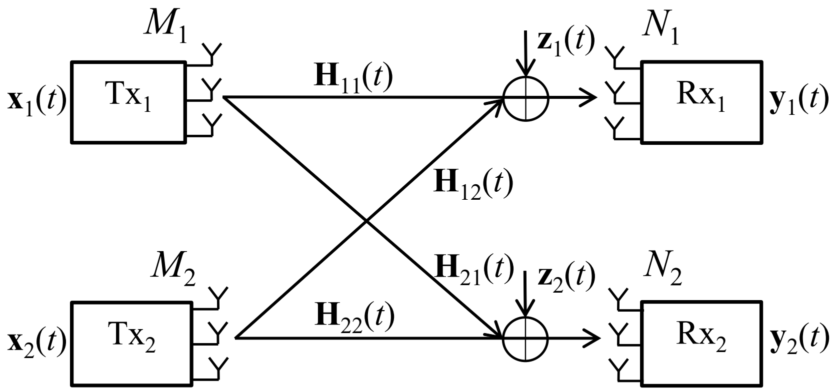

Figure 2.1: The MIMO X-Channel

In this chapter, we consider linear coding strategies, in which the achievable DoF are given by the dimension of the useful signal space normalized with respect to the communication duration [BCT14]. We assume that during the communication duration n, Tx_i, $i \in \{1,2\}$, intends to deliver a $b_{ji}(n)$-element symbol vector $\mathbf{u}_{ji} \in \mathbb{C}^{b_{ji}(n)\times 1}$ to Rx_j. The signal transmitted by Tx_i is linearly precoded, where at the t-th channel use the transmitted signal is given by

$$\mathbf{x}_i(t) = \mathbf{C}_{1i}(t)\,\mathbf{u}_{1i} + \mathbf{C}_{2i}(t)\,\mathbf{u}_{2i}, \tag{2.3}$$

with $\mathbf{C}_{ji}(t) \in \mathbb{C}^{M_i \times b_{ji}(n)}$ being the precoding matrix. By denoting the vertical concatenation of the transmitted signals and precoding matrices as $\mathbf{x}_i^n \in \mathbb{C}^{nM_i\times 1}$ and $\mathbf{C}_{ji}^n \in \mathbb{C}^{nM_i\times b_{ji}(n)}$, respectively, the vertical concatenation of the signals received by Rx_j can be evaluated as

$$\mathbf{y}_j^n = \mathbf{H}_{j1}^n\left(\mathbf{C}_{11}^n\mathbf{u}_{11} + \mathbf{C}_{21}^n\mathbf{u}_{21}\right) + \mathbf{H}_{j2}^n\left(\mathbf{C}_{12}^n\mathbf{u}_{12} + \mathbf{C}_{22}^n\mathbf{u}_{22}\right) + \mathbf{z}_j^n \in \mathbb{C}^{nN_j\times 1}, \tag{2.4}$$

where $\mathbf{H}_{ji}^n \in \mathbb{C}^{nN_j\times nM_i}$ is a diagonal concatenation of the channel matrices between Tx_i and Rx_j, and $\mathbf{z}_j^n \in \mathbb{C}^{nN_j\times 1}$ is the vertical concatenation of noise vectors.

Following [LAS14], we introduce the condition on the decodability for $\mathbf{u}_{ji}$. Let

$$\mathcal{I}_{ji} = \mathrm{span}\left(\bigcup_{1\le k,l\le 2,(k,l)\ne(i,j)} \mathbf{H}_{lk}^n\mathbf{C}_{lk}^n\right) \subseteq \mathbb{C}^{nN_j\times 1} \tag{2.5}$$

denote the subspace at Rx_j containing the interference signal, where $\mathrm{span}(\mathbf{A}) = \left\{\mathbf{A}\mathbf{x}\,\middle|\,\mathbf{x}\in\mathbb{C}^{k\times 1}\right\}$, is a space spanned by the columns of $\mathbf{A} \in \mathbb{C}^{l\times k}$. The interference-free subspace is then given by $\mathcal{I}_{ji}^{\mathrm{c}} \subseteq \mathbb{C}^{nN_j\times 1}$, where $\mathcal{A}^{\mathrm{c}} \subseteq \mathbb{C}^{l\times 1}$ denotes the subspace complementary to $\mathcal{A} \subseteq \mathbb{C}^{l\times 1}$. Given two subspaces $\mathcal{A},\mathcal{B} \subseteq \mathbb{C}^{l\times 1}$, the projection of $\mathcal{B}$ on $\mathcal{A}$ is given by $\mathrm{proj}_{\mathcal{A}}\mathcal{B} = \left\{\mathbf{x}\in\mathcal{A}\,\middle|\,\exists\mathbf{y}\in\mathcal{B},\ \text{s.t. } \mathbf{x}^{\mathrm{H}}\mathbf{y}\ne 0\right\} \subseteq \mathbb{C}^{l\times 1}$. In the limit of large

P where noise has no influence on the decodability, the vector of information symbols $\mathbf{u}_{ji}$ is said to be decodable if

$$\dim\left(\mathrm{proj}_{\mathcal{I}_{ji}^c}\mathrm{span}\left(\mathbf{H}_{ji}^n\mathbf{C}_{ji}^n\right)\right) = b_{ji}\left(n\right), \tag{2.6}$$

where $\dim\left(.\right)$ is the dimension of the subspace.

Further following [LAS14], we introduce the definition of the linear DoF as follows. The DoF tuple $(d_{11}, d_{12}, d_{21}, d_{22})$ is deemed to be linearly achievable if there exists a sequence of precoding matrix tuples $(\mathbf{C}_{11}^n, \mathbf{C}_{12}^n, \mathbf{C}_{21}^n, \mathbf{C}_{22}^n)$ fulfilling the decodability condition (3.6) for which

$$d_{ji} = \lim_{n\to\infty}\frac{b_{ji}\left(n\right)}{n} \tag{2.7}$$

holds. The closure of all linearly achievable DoF tuples $(d_{11}, d_{12}, d_{21}, d_{22})$ is called the DoF region $\mathcal{D}_{\mathrm{lin}}$. The linearly achievable sum-DoF is denoted by $d_\Sigma = d_{11} + d_{12} + d_{21} + d_{22}$. The maximum linearly achievable sum-DoF (or simply linear DoF) is defined as

$$d_{\mathrm{lin}} = \max_{(d_{11},d_{12},d_{21},d_{22})\in\mathcal{D}_{\mathrm{lin}}} d_{11} + d_{12} + d_{21} + d_{22}. \tag{2.8}$$

2.3 Achievability Scheme

In this section, the transmission scheme given in [KA17] for the antenna configurations $N_1 < M_1 + M_2$, $M_1 < N_1 + N_2$ and $M_2 > N_2$ is given in detail.

2.3.1 Overview

The transmission comprises three phases referred to as phases 1, 2 and 3. Phase 1 is dedicated to the transmission of information symbols to Rx_1, while Rx_2 overhears interference in phase 1. In turn, phase 2 is dedicated to the transmission of information symbols to Rx_2, while Rx_1 overhears interference in phase 2. From the interference terms overheard at the unintended receivers in phases 1 and 2, terms useful for a single receiver and known at another single receiver, referred to as order-(1,1) symbols, are generated. The delivery of the order-(1,1) symbols is performed in phase 3 where order-(1,1) symbols generated in phases 1 and 2 are transmitted simultaneously.

Depending on the antenna configuration, different transmission approaches are applied in phases 1 and 2. Since we initially assumed that $M_1 \geq M_2 > N_2$ holds, in phase 1

IS-RT-PIN is applied, where Tx_1 uses all available transmit dimensions and Tx_2 applies IS-RT. The transmission in phase 2 depends on the relationship between M_1, M_2 and N_1. As given in [KA17], the following three cases are distinguished.

1. $M_1 \geq M_2 > N_1$

   As in phase 1, IS-RT-PIN is applied, where Tx_1 uses all available transmit dimensions and Tx_2 applies IS-RT.

2. $M_1 > N_1 \geq M_2$

   RT-PIN is applied, where Tx_1 uses all available transmit dimensions and Tx_2 applies RT.

3. $N_1 \geq M_1 \geq M_2$

   RT-PIN is applied, where Tx_1 and Tx_2 apply RT.

2.3.2 Transmission Blocks

For the description of phases 1 and 2, the notion of transmission blocks will be used. Transmission blocks are transmission periods with identical structure, but different transmitted information symbols. Phases 1 and 2 are comprised of multiple transmission blocks, where the structure of a single transmission block is designed for each phase independently. After the transmission blocks are designed, phases 1 and 2 are balanced, where the numbers of transmission blocks are chosen to ensure that all generated order-(1,1) symbols can be delivered in phase 3 to the receivers which desire them.

Phase l, $l \in \{1, 2\}$, is comprised of $k^{(l)}$ transmission blocks having duration $T^{(l)}$, where the total duration of phase l is $T_{\Sigma}^{(2)} = k^{(2)} T^{(2)}$ channel uses. During the transmission block of phase 1, Tx_1 transmits $b_{11}^{(1)}$ information symbols and Tx_2 transmits $b_{12}^{(1)}$ information symbols. During the transmission block of phase 2, Tx_1 transmits $b_{21}^{(1)}$ information symbols and Tx_2 transmits $b_{22}^{(1)}$ information symbols. After the transmission of a transmission block of phase l, $l \in \{1, 2\}$, $q^{(l)}$ order-(1,1) symbols to be transmitted in phase 3 are generated. The delivery of the generated order-(1,1) symbols is performed in phase 3 having a total duration of $T_{\Sigma}^{(3)}$ channel uses.

In a single channel use of phase 3, $\min\{M_1, N_1\}$ order-(1,1) symbols can be delivered to Rx_1 and N_2 order-(1,1) symbols can be delivered to Rx_2. To ensure that all order-(1,1) symbols generated in phases 1 and 2 can be delivered to Rx_1 and Rx_2 in phase 3, the numbers of the transmission blocks of phases 1 and 2 are chosen to fulfil

$$\frac{k^{(1)}q^{(1)}}{N_1} = \frac{k^{(2)}q^{(2)}}{N_2}, \tag{2.9}$$

for $M_1 \geq N_1$ and

$$\frac{k^{(1)}q^{(1)}}{M_1} = \frac{k^{(2)}q^{(2)}}{N_2}, \tag{2.10}$$

for $M_1 < N_1$.

In the next section, we will describe the transmission in phases 1 and 2 in order to perform the decodability analysis later. Since the transmission relying on RT-PIN is not relevant for the decodability analysis, it will be omitted from the further description and only transmission relying on IS-RT-PIN will be described. Due to symmetry, we will describe the transmission relying on IS-RT-PIN only for phase 1 and the transmission for phase 2 can be obtained by swapping the receivers' indices.

2.3.3 Phase 1

In this section, we describe the transmission in phase 1 as given in [KA17] which relies on IS-RT-PIN.

The transmission block of phase 1 is split into parts 1 and 2, where the duration of part l, $l \in \{1, 2\}$ is denoted by $T^{(1,l)}$, with

$$T^{(1,1)} + T^{(1,2)} = T^{(1)}. \tag{2.11}$$

In parts 1 and 2, the transmission is performed as follows.

- Part 1 (IS): both transmitters transmit new information symbols, where the interference signal of Tx_2 overheard by Rx_2 comprieses the redundancy to be transmitted by Tx_2 in part 2.
- Part 2 (RT): Tx_1 continues to transmit new information symbols where Tx_2 transmits the redundancy generated in part 1.

In the following, the transmission in parts 1 and 2 will be given in detail. Then, the generation of order-(1,1) symbols will be described.

Part 1 (IS): In part 1, Tx_1 transmits the symbol vector $\mathbf{u}_{11}^{(1,1)} \in M_1 T^{(1,1)}$ and Tx_2 transmits the symbol vector $\mathbf{u}_{12}^{(1)} \in b_{12}^{(1)}$. In each channel use, Tx_1 transmits a new symbol from each antenna and Tx_2 applies random precoding. The vertical concatenations of the signals transmitted by Tx_1 and Tx_2 are given by

$$\mathbf{x}_1^{(1,1)} = \mathbf{u}_{11}^{(1,1)} \in \mathbb{C}^{M_1 T^{(1,1)} \times 1}, \tag{2.12}$$

$$\mathbf{x}_2^{(1,1)} = \mathbf{C}_{12}^{(1,1)} \mathbf{u}_{12}^{(1)} \in \mathbb{C}^{M_2 T^{(1,1)} \times 1}, \tag{2.13}$$

where $\mathbf{C}_2^{(1,1)} \in \mathbb{C}^{M_2 T^{(1,1)} \times b_{12}^{(1)}}$ is a matrix of random precoding coefficients. By omitting the receive noise term, the vertical concatenation of the signals received by Rx_j, $j \in \{1,2\}$, in part 1 is evaluated as

$$\mathbf{y}_j^{(1,1)} = \mathbf{H}_{j1}^{(1,1)} \mathbf{u}_{11}^{(1,1)} + \mathbf{H}_{j2}^{(1,1)} \mathbf{C}_{12}^{(1,1)} \mathbf{u}_{12}^{(1)} \in \mathbb{C}^{N_j T^{(1,1)} \times 1}, \tag{2.14}$$

where $\mathbf{H}_{ji}^{(1,1)} \in \mathbb{C}^{N_j T^{(1,1)} \times M_i T^{(1,1)}}$ is the diagonal concatenation of the channel matrices between Tx_i and Rx_j in part 1.

Part 2 (RT): In part 2, Tx_1 transmits the symbol vector $\mathbf{u}_{11}^{(1,2)} \in M_1 T^{(1,2)}$ and Tx_2 retransmits the interference signals overheard by Rx_2 in part 1. The signals transmitted by Tx_1 and Tx_2 are given by

$$\mathbf{x}_1^{(1,2)} = \mathbf{u}_{11}^{(1,2)} \in \mathbb{C}^{M_1 T^{(1,2)} \times 1}, \tag{2.15}$$

$$\mathbf{x}_2^{(1,2)} = \mathbf{C}_{12}^{(1,2)} \mathbf{H}_{22}^{(1,1)} \mathbf{C}_2^{(1,1)} \mathbf{u}_{12}^{(1)} \in \mathbb{C}^{M_2 T^{(1,2)} \times 1}, \tag{2.16}$$

where $\mathbf{C}_{12}^{(1,2)} \in \mathbb{C}^{M_2 T^{(1,2)} \times N_2 T^{(1,1)}}$ is the matrix of random precodig coefficients. The signal received by Rx_j in part 2 is evaluated as

$$\mathbf{y}_j^{(1,2)} = \mathbf{H}_{j1}^{(1,2)} \mathbf{u}_{11}^{(1,2)} + \mathbf{H}_{j2}^{(1,2)} \mathbf{C}_{12}^{(1,2)} \mathbf{H}_{22}^{(1,2)} \mathbf{C}_{12}^{(1,1)} \mathbf{u}_{12}^{(1)} \in \mathbb{C}^{N_j T^{(1,2)} \times 1}, \tag{2.17}$$

where $\mathbf{H}_{ji}^{(1,2)} \in \mathbb{C}^{N_j T^{(1,2)} \times M_i T^{(1,2)}}$ is the diagonal concatenation of the channel matrices between Tx_i and Rx_j in part 2.

PIN: For the order-(1,1) symbol generation, Rx_2 applies PIN to cancel the signal transmitted by Tx_2 from the received signal. Below we describe the PIN procedure in detail.

First, let us collect the signals transmitted by Tx_1 into the vector

$$\mathbf{u}_{11}^{(1)} = \begin{bmatrix} \mathbf{u}_{11}^{(1,1)} \\ \mathbf{u}_{11}^{(1,2)} \end{bmatrix} \in \mathbb{C}^{M_1 T^{(1)} \times 1} \tag{2.18}$$

and the signal received by Rx_j, $j \in \{1,2\}$, into the vector

$$\mathbf{y}_j^{(1)} = \begin{bmatrix} \mathbf{y}_j^{(1,1)} \\ \mathbf{y}_j^{(1,2)} \end{bmatrix} \in \mathbb{C}^{N_j T^{(1)} \times 1}. \tag{2.19}$$

The signal vector $\mathbf{y}_j^{(1)}$ can then be written in the form

$$\mathbf{y}_j^{(1)} = \bar{\mathbf{H}}_{j1}^{(1)}\mathbf{u}_{11}^{(1)} + \bar{\mathbf{H}}_{j2}^{(1)}\mathbf{u}_{12}^{(1)}, \tag{2.20}$$

where $\bar{\mathbf{H}}_{j1}^{(1)} \in \mathbb{C}^{N_jT^{(1)}\times M_1T^{(1)}}$ and $\bar{\mathbf{H}}_{j2}^{(1)} \in \mathbb{C}^{N_jT^{(1)}\times M_2T^{(1)}}$ are the effective channel matrices given by

$$\bar{\mathbf{H}}_{j1}^{(1)} = \begin{bmatrix}\mathbf{H}_{j1}^{(1,1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{j1}^{(1,2)}\end{bmatrix}, \ \bar{\mathbf{H}}_{j2}^{(1)} = \begin{bmatrix}\mathbf{H}_{j2}^{(1,1)}\mathbf{C}_{12}^{(1,1)}\\ \mathbf{H}_{j2}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{22}^{(1)}\mathbf{C}_{12}^{(1,1)}\end{bmatrix}. \tag{2.21}$$

Let us consider the signal $\mathbf{y}_2^{(1)}$ received by Tx_2. The effective channel matrix $\bar{\mathbf{H}}_{22}^{(1)}$ can be written in the form

$$\bar{\mathbf{H}}_{22}^{(1)} = \begin{bmatrix}\mathbf{I}_{N_2T^{(1)}}\\ \mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\end{bmatrix}\mathbf{H}_{22}^{(1,1)}\mathbf{C}_{12}^{(1,1)}, \tag{2.22}$$

which is full rank almost surely. The size of the left null space of $\bar{\mathbf{H}}_{22}^{(1)}$ is given by $N_2T^{(1,2)}$, hence there exists a full rank matrix $\mathbf{W}_{12}^{(1)} \in \mathbb{C}^{N_2T^{(1)}\times N_2T^{(1,2)}}$, for which

$$\mathbf{W}_{22}^{(1)\text{H}}\begin{bmatrix}\mathbf{I}_{N_2T^{(1)}}\\ \mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\end{bmatrix} = \mathbf{0}_{N_2T^{(1,2)}\times N_2T^{(1)}} \tag{2.23}$$

holds. Without loss of generality, we assume $\mathbf{W}_{12}^{(1)}$ to have the form of

$$\mathbf{W}_{22}^{(1)} = \begin{bmatrix}-\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1)} & \mathbf{I}_{N_2T^{(1,2)}}\end{bmatrix}. \tag{2.24}$$

By projecting the received signal $\mathbf{y}_2^{(1)}$ onto $\mathbf{W}_{22}^{(1)}$, the term containing only the signal of Tx_1,

$$\mathbf{W}_{22}^{(1)\text{H}}\mathbf{y}_2^{(1)} = \begin{bmatrix}-\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{21}^{(1,1)} & \mathbf{H}_{21}^{(1,2)}\end{bmatrix}\mathbf{u}_{11}^{(1)} \in \mathbb{C}^{N_2T^{(1,2)}\times 1}, \tag{2.25}$$

is obtained. We refer to the obtained term $\mathbf{W}_{22}^{(1)\text{H}}\mathbf{y}_2^{(1)} = \mathbf{u}_{1|1;2}$ as a vector of order-(1,1) symbols which are desired by Rx_1 and known at Rx_2.

2.4 Decodability Analysis

In this section, the decodability analysis of the transmission scheme proposed by [KA17] is performed. As we already mentioned, we perform the decodability analysis of the symbols transmitted using IS-RT-PIN. Due to the symmetry of phases 1 and 2, for the decodability analysis we consider the symbols transmitted in phase 1 and the results of the decodability analysis can be extended to phase 2 by swapping the receivers' indices. For our study, we consider the linear combinations provided to Rx_1 by the signal $\mathbf{y}_1^{(1)}$

and the order-(1,1) symbol vector $\mathbf{u}_{1|1;2}$. The details of the decodability analysis are provided below.

Firstly, we assume that Rx_1 obtains the number of linear combinations sufficient for decoding, which is ensured by setting the number of information symbols transmitted by Tx_2 as

$$b_{12}^{(1)} = (N_1 + N_2 - M_1)\, T^{(1)} - N_2 T^{(1,1)}, \tag{2.26}$$

expressing thus $b_{12}^{(1)}$ as a function of $T^{(1,1)}$ and $T^{(1)}$. $b_{12}^{(1)}$ however cannot exceed the number of the transmit dimensions of Tx_2, hence

$$b_{12}^{(1)} \leq M_2 T^{(1,1)} \tag{2.27}$$

is a necessary condition to be ensured for the decodability. Plugging (2.26) into (2.27) gives us the decodability bound on $\frac{T^{(1,1)}}{T^{(1)}}$ as

$$B_1 \equiv \frac{T^{(1,1)}}{T^{(1)}} \geq \frac{N_1 + N_2 - M_1}{M_2 + N_2}. \tag{2.28}$$

Maximizing $\frac{b_{11}^{(1)} + b_{12}^{(1)}}{T^{(1)}}$ is equivalent to maximizing $\frac{b_{12}^{(1)}}{T^{(1)}}$, which in turn is equivalent to minimizing $\frac{T^{(1,1)}}{T^{(1)}}$. Hence, without any additional constrains on the decodability, $\frac{T^{(1,1)}}{T^{(1)}}$ has to be chosen as a minimum satisfying B_1 in order to maximize the achievable DoF. This will result in the parameters of the transmission scheme in [KA17]. To show the loss of the decodability in [KA17], additional decodability bounds have to be provided which override B_1. Before providing additional decodability bounds, we first give a motivating example demonstrating the loss of decodability by choosing the parameters of the transmission as in [KA17].

Example: Suppose $M_1 = M_2 = 4$, $N_1 = 6$ and $N_2 = 1$, where according to [KA17], $T = 5$, $T^{(1,1)} = 3$ and $T^{(1,2)} = 2$. In part 2 of the transmission block, Tx_1 transmits $M_1 T^{(1,2)} = 8$ symbols and Tx_2 retransmits $N_2 T^{(1,1)} = 3$ terms. The length of $\mathbf{y}_1^{(1,2)}$ is $N_1 T^{(1,2)} = 12 > M_1 T^{(1,2)} + N_2 T^{(1)} = 11$, hence the linear combinations comprising $\mathbf{y}_1^{(1,2)}$ are linearly dependent. Having a linearly dependent subset of linear combinations used for decodability results in a loss of decodability.

To obtain new bounds on the decodability, we perform linear independence analysis of the linear combinations comprising $\mathbf{y}_1^{(1)}$ and $\mathbf{u}_{1|1;2}$. First, let us construct the effective channel matrix, the rows of which comprise the linear combinations of the elements of $\mathbf{u}_{11}^{(1)}$ and $\mathbf{u}_{12}^{(1)}$. By concatenating $\mathbf{y}_1^{(1)}$ and $\mathbf{u}_{1|1;2}$ we have

$$\begin{bmatrix} \mathbf{y}_1^{(1)} \\ \mathbf{u}_{1;2} \end{bmatrix} = \bar{\mathbf{H}}_1 \begin{bmatrix} \mathbf{u}_{11}^{(1)} \\ \mathbf{u}_{12}^{(1)} \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^{\left(N_1 T^{(1)} + N_2 T^{(1,2)}\right) \times 1}, \tag{2.29}$$

where $\bar{\mathbf{H}}_1 \in \mathbb{C}^{\left(N_1 T^{(1)}+N_2 T^{(1,2)}\right)\times\left(N_1 T^{(1)}+N_2 T^{(1,2)}\right)}$ is the effective channel matrix given by

$$\bar{\mathbf{H}}_1 = \begin{bmatrix} \mathbf{H}_{11}^{(1,1)} & \mathbf{0} & \mathbf{H}_{12}^{(1,1)}\mathbf{C}_{12}^{(1,1)} \\ \mathbf{0} & \mathbf{H}_{11}^{(1,2)} & \mathbf{H}_{12}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{22}^{(1,1)}\mathbf{C}_{12}^{(1,1)} \\ -\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{21}^{(1,1)} & \mathbf{H}_{21}^{(1,2)} & \mathbf{0} \end{bmatrix}. \tag{2.30}$$

The following lemma establishes the condition when $\bar{\mathbf{H}}_1$ is rank deficient.

Lemma 1. $\bar{\mathbf{H}}_1$ *is rank deficient if*

$$N_2 \min\left\{T, 2T^{(1)}\right\} + M_1 T^{(2)} < \left(N_1 + N_2\right) T^{(2)}. \tag{2.31}$$

Proof. Let us consider the matrix which is comprised of the last $N_1 T^{(1,2)} + N_2 T^{(1,2)}$ rows of $\bar{\mathbf{H}}_1$:

$$\bar{\mathbf{H}}_1' = \begin{bmatrix} \mathbf{0} & \mathbf{H}_{11}^{(1,2)} & \mathbf{H}_{12}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{22}^{(1,1)}\mathbf{C}_{12}^{(1,1)} \\ -\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{21}^{(1,1)} & \mathbf{H}_{21}^{(1,2)} & \mathbf{0} \end{bmatrix}. \tag{2.32}$$

Given $\bar{\mathbf{H}}_1'$ is rank deficient, $\bar{\mathbf{H}}_1$ is rank deficient as well. $\bar{\mathbf{H}}_1'$ has at most a rank of

$$\operatorname{rank}\left(\bar{\mathbf{H}}_1'\right) \leq \left(N_1 + N_2\right) T^{(1,2)}, \tag{2.33}$$

where to show rank deficiency of $\bar{\mathbf{H}}_1'$, it suffices to find an upper bound on the rank of $\bar{\mathbf{H}}_1'$ overriding (2.33).

First, using the rank property of horizontally concatenated matrices, we upper bound the rank of $\mathbf{H}_1'$ by the sum of the ranks of the matrices constituting it as

$$\operatorname{rank}\left(\mathbf{H}_1'\right) \leq$$
$$\operatorname{rank}\left(\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{21}^{(1,1)}\right) + \operatorname{rank}\left(\begin{bmatrix} \mathbf{H}_{11}^{(1,2)} \\ \mathbf{H}_{21}^{(1,2)} \end{bmatrix}\right) + \operatorname{rank}\left(\mathbf{H}_{12}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{22}^{(1,1)}\mathbf{C}_{12}^{(1,1)}\right). \tag{2.34}$$

The terms constituting the right hand side of (2.34) can be upper bounded as

$$\operatorname{rank}\left(\mathbf{H}_{22}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{21}^{(1,1)}\right) \leq N_2 \min\left\{T^{(1,1)}, T^{(1,2)}\right\}, \tag{2.35}$$

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{H}_{11}^{(1,2)} \\ \mathbf{H}_{21}^{(1,2)} \end{bmatrix}\right) \leq M_1 T^{(1,2)} \tag{2.36}$$

$$\operatorname{rank}\left(\mathbf{H}_{12}^{(1,2)}\mathbf{C}_{12}^{(1,2)}\mathbf{H}_{22}^{(1,1)}\mathbf{C}_{12}^{(1,1)}\right) \leq \min\left\{N_1 T^{(1,2)}, M_2 T^{(1,2)}, N_2 T^{(1,1)}\right\}, \tag{2.37}$$

where for (2.35) and (2.37), we used the rank property of matrix products. By inserting (2.35), (2.36) and (2.37) in (2.34) and using the property (2.33), one obtains

$$\operatorname{rank}\left(\bar{\mathbf{H}}_1'\right) \leq N_2 \min\left\{T, 2T^{(1,1)}\right\} + M_1 T^{(1,2)}, \tag{2.38}$$

which overrides (2.33) when (2.31) holds.

□

The condition (2.31) can be rewritten as the following two bounds on $\frac{T^{(1,1)}}{T^{(1)}}$:

$$B_2 \equiv \frac{T^{(1,1)}}{T^{(1)}} \geq \frac{N_1 + N_2 - M_1}{N_1 + 3N_2 - M_1}, \tag{2.39}$$

$$B_3 \equiv \frac{T^{(1,1)}}{T^{(1)}} \geq \frac{N_1 - M_1}{N_1 + N_2 - M_1}. \tag{2.40}$$

The cases when B_2 and B_3 override B_1 and the choice of $T^{(1,1)}$ and $T^{(1)}$ to maximize $b_{12}^{(1)}$ while ensuring decodability will be given in the next section.

2.5 Proposed Transmission Scheme

In this section, we describe the proposed transmission scheme, which is obtained by modifying the transmission scheme of [KA17], where we adjust the parameters of the transmission relying on IS-RT-PIN to maximize the achieved number of DoF while avoiding the loss of decodability identified in the previous section. We describe the choice of the parameters for phase 1, and the choise of parameters for phase 2 can be obtained by swapping the receivers' indices.

As already mentioned, maximizing the achievable DoF is equivalent to minimizing $\frac{T^{(1,1)}}{T^{(1)}}$. In order to avoid the loss of decodability, we propose to choose $\frac{T^{(1,1)}}{T^{(1)}}$ as a minimum satisfying B_1, B_2 and B_3. Depending on whether B_1, B_2 or B_3 are active, the following three regions of antenna configurations are distinguished.

Region 1 (B_1 is active): In this region B_1, overrides B_2 and B_3. The region of antenna configurations is given by

$$\begin{aligned} M_1 + M_2 &\leq N_1 + 2N_2, \\ (N_1 + N_2 - M_1)^2 &\geq (M_2 + N_2)(N_1 - M_1). \end{aligned} \tag{2.41}$$

To maximize $\frac{b_{12}^{(1)}}{T^{(1)}}$ while ensuring B_1, we choose $T^{(1,1)}$ and $T^{(1,1)}$ as

$$T = M_2 + N_2, \; T^{(1)} = N_1 + N_2 - M_1, \tag{2.42}$$

where from (2.26), $b_{12}^{(1)} = M_2 (N_1 + N_2 - M_1)$ follows. In Region 1, the parameters of the transmission in phase 1 are identical to that of [KA17].

Region 2 (B_1 is active): In this region B_2, overrides B_1 and B_3. The region of antenna configurations is given by

$$\begin{aligned} M_1 + M_2 &> N_1 + 2N_2, \\ N_1 - M_1 &\leq N_2. \end{aligned} \tag{2.43}$$

To maximize $\frac{b_{12}^{(1)}}{T^{(1)}}$ while ensuring B_2, we choose $T^{(1,1)}$ and $T^{(1,1)}$ as

$$T^{(1)} = N_1 + 3N_2 - M_1,\ T^{(1,1)} = N_1 + N_2 - M_1, \tag{2.44}$$

where from (2.26), $b_{12}^{(1)} = (N_1 + 2N_2 - M_1)(N_1 + N_2 - M_1)$ follows. In Region 2, the information symbols transmitted using the transmission scheme [KA17] are not decodable.

Region 3 (B_3 is active): In this region B_3, overrides B_1 and B_2. The region of antenna configurations is given by

$$\begin{aligned}(N_1 + N_2 - M_1)^2 &< (M_2 + N_2)(N_1 - M_1), \\ N_1 - M_1 &> N_2.\end{aligned} \tag{2.45}$$

To maximize $\frac{b_{12}^{(1)}}{T^{(1)}}$ while ensuring B_2, we choose $T^{(1,1)}$ and $T^{(1,1)}$ as

$$T = N_1 + N_2 - M_1,\ T^{(1)} = N_1 - M_1, \tag{2.46}$$

where from (2.26), $b_{12}^{(1)} = (N_1 + N_2 - M_1)^2 - N_2(N_1 - M_1)$ follows. In Region 3, the information symbols transmitted using the transmission scheme [KA17] are not decodable.

The obtained regions of the antenna configurations for the cases when $N_1 = N_2 = N$ and $M_1 = M_2 = M$ are illustrated in Figure 2.2.

2.6 Achieved Number of DoF

In this section, we calculate the DoF achieved by the proposed transmission scheme and compare it to the DoF which can be achieved by the transmission scheme of [KA17] in which the number of the transmitted information symbols is reduced to the number of the decodable ones.

As previously mentioned, in order maximize to the achievable DoF, phase 1 and 2 are to be balanced, such that (2.9) and (2.10) hold for $M_1 \geq N_1$ and $M_1 < N_1$, respectively. For $M_1 \geq N_1$, we set

$$k^{(1)} = q^{(2)}N_1,\ k^{(2)} = q^{(1)}N_2. \tag{2.47}$$

Then, by using the fact that

$$q^{(1)} = b_{11}^{(1)} + b_{11}^{(1)} - T^{(1)}N_1, \tag{2.48}$$

$$q^{(2)} = b_{21}^{(2)} + b_{21}^{(2)} - T^{(2)}N_2, \tag{2.49}$$

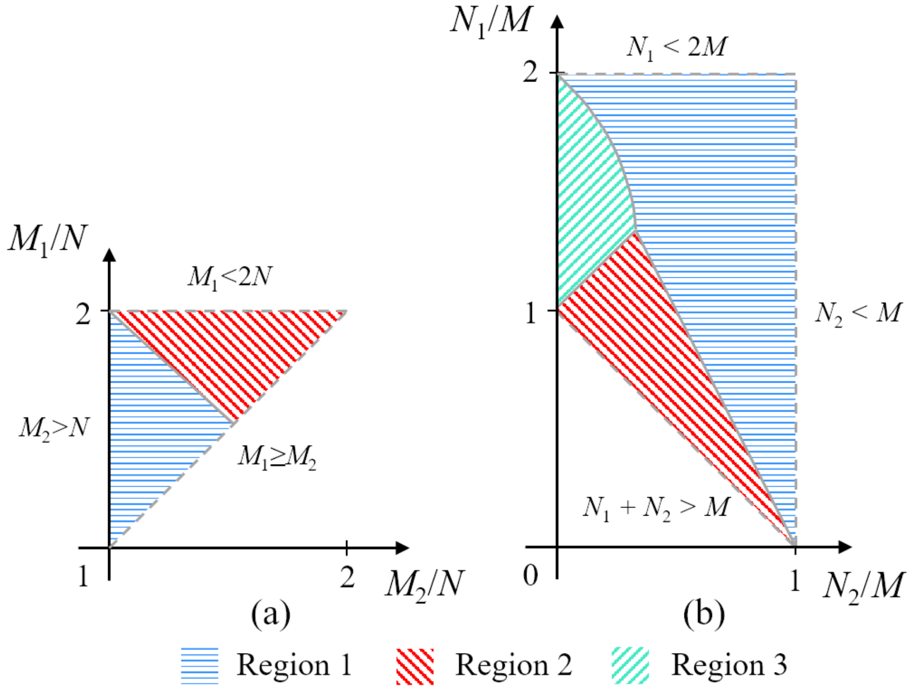

Figure 2.2: Region 1, 2 and 3 for the cases of (a) $N_1 = N_2 = N$ and (b) $M_1 = M_2 = M$.

the achieved DoF can be expressed following the derivation in [KA17] as

$$d_\Sigma = \frac{\Gamma_1 \Gamma_2 (N_1 + N_2) - \Gamma_1 N_2 - \Gamma_2 N_1}{\Gamma_1 \Gamma_2 - 1}, \tag{2.50}$$

where

$$\Gamma_1 = \frac{b_{11}^{(1)} + b_{11}^{(1)}}{N_2 T^{(1)}}, \tag{2.51}$$

$$\Gamma_2 = \frac{b_{21}^{(2)} + b_{21}^{(2)}}{N_1 T^{(2)}}, \tag{2.52}$$

are the ratios of the sizes of the useful to interference signal spaces for the information symbols transmitted to Rx_1 and Rx_2, respectively.

For $M_1 < N_1$, we set

$$k^{(1)} = q^{(2)} M_1, \; k^{(2)} = q^{(1)} N_2. \tag{2.53}$$

Then, the achieved DoF can be evaluated as

$$d_\Sigma = \frac{\Gamma_1 \Gamma_2^* (M_1 + N_2) - \Gamma_1 N_2 - \Gamma_2^* N_1}{\left(\Gamma_1 - \frac{N_1 - M_1}{N_2}\right) \Gamma_2^* - 1}, \tag{2.54}$$

where Γ_1 is given by (2.51) and Γ_2^* is the factor evaluated as

$$\Gamma_2^* = \frac{b_{21}^{(2)} + b_{21}^{(2)}}{M_1 T^{(2)}}. \tag{2.55}$$

The calculations of factors Γ_1, Γ_2 and Γ_2^* for the cases $M_1 \geq M_2 > N_1$, $M_1 > N_1 \geq M_2$ and $N_1 \geq M_1 \geq M_2$ are summarized in Tables 2.1, 2.2 and 2.3.

Comparison to the LDoF upper bound [KA17]: [KA17] initially claimed the achievability of the upper bound for all antenna configurations. For the case $M_1 \geq N_1$ and $N_1 + 2N_2 \geq M_1 + M_2$, the proposed transmission scheme is identical to that in [KA17], hence it is LDoF optimal. For the remaining cases, the proposed transmission scheme achieves a number of DoF less than that initially claimed in [KA17], hence no LDoF optimality can be stated.

In the following, we calculate the achieved number of DoF for the symmetric antenna configurations where $M_1 = M_2 = M$ and $N_1 = N_2 = N$ hold. In this case, phases 1 and 2 have identical parameters, where Region 3 is empty and Regions 1 and 2 correspond to $1 < \frac{M}{N} \leq \frac{3}{2}$ and $\frac{3}{2} < \frac{M}{N} < 2$, respectively. By referring to (2.50), we calculate the achieved DoF as

$$d = \begin{cases} \frac{6MN}{4M+N}, & \text{if } 1 < \frac{M}{N} \leq \frac{3}{2}, \\ \frac{N(6N-M)}{5N-M}, & \text{if } \frac{3}{2} < \frac{M}{N} < 2, \end{cases} \tag{2.56}$$

which is identical to that of [KA17] for $1 < \frac{M}{N} \leq \frac{3}{2}$.

In the region of $\frac{3}{2} < \frac{M}{N} < 2$, we compare the proposed transmission scheme to the transmission scheme of [KA17], where the number of the transmitted information symbols is reduced to the number of the decodable ones. For phases 1, we calculate the number of the decodable information symbols using the left hand side of (2.31) as

$$b_{11}^{(1)} + b_{12}^{(1)} = MT^{(1)} + (3N - M)\, T^{(1,1)}, \tag{2.57}$$

where

$$T^{(1,1)} = 2N - M,\ T^{(1)} = M + N. \tag{2.58}$$

The ratios of useful to interference signal spaces are then calculated as

$$\Gamma_1 = \Gamma_2 = \frac{2M^2 - 4MN + 6N^2}{N\,(M+N)}, \tag{2.59}$$

where the achieved number of DoF is finally evaluated using (2.50) as

$$d_\Sigma = \frac{2N\,(2M^2 - 4MN + 6N^2)}{2M^2 - 3MN + 7N^2}. \tag{2.60}$$

Table 2.1: Summary of Γ_1 and Γ_2 for $M_1 \geq M_2 > N_1$

Region	Γ_1	Γ_2
$N_1 + 2N_2 \geq M_1 + M_2$	$\frac{M_1(M_2+N_2)+M_2(N_1+N_2-M_1)}{N_2(M_2+N_2)}$	$\frac{M_1(M_2+N_1)+M_2(N_1+N_2-M_1)}{N_1(M_2+N_1)}$
$2N_1 + N_2 \geq M_1 + M_2 > N_1 + 2N_2$	$\frac{M_1(N_1+3N_2-M_1)+(N_1+2N_2-M_1)(N_1+N_2-M_1)}{N_2(N_1+N_2-M_1)}$	$\frac{M_1(M_2+N_1)+M_2(N_1+N_2-M_1)}{N_1(M_2+N_1)}$
$M_1 + M_2 > 2N_1 + N_2$	$\frac{M_1(N_1+3N_2-M_1)+(N_1+2N_2-M_1)(N_1+N_2-M_1)}{N_2(N_1+N_2-M_1)}$	$\frac{M_1(3N_1+N_2-M_1)+(2N_1+N_2-M_1)(N_1+N_2-M_1)}{N_1(N_1+N_2-M_1)}$

Table 2.2: Summary of Γ_1 and Γ_2 for $M_1 > N_1 \geq M_2$

Region	Γ_1	Γ_2
$N_1 + 2N_2 \geq M_1 + M_2$	$\frac{M_1(M_2+N_2)+M_2(N_1+N_2-M_1)}{N_2(M_2+N_2)}$	$\frac{M_1+N_1+N_2}{2N_1}$
$M_1 + M_2 > N_1 + 2N_2$	$\frac{M_1(N_1+3N_2-M_1)+(N_1+2N_2-M_1)(N_1+N_2-M_1)}{N_2(N_1+N_2-M_1)}$	$\frac{M_1+N_1+N_2}{2N_1}$

Table 2.3: Summary of Γ_1 and Γ_2^* for $N_1 \geq M_1 \geq M_2$

Region	Γ_1	Γ_2^*
$N_1 + \frac{N_2}{2} \geq M_1 + M_2$	$\frac{M_1(M_2+N_2)+M_2(N_1+N_2-M_1)}{N_2(M_2+N_2)}$	$\frac{M_1+M_2}{M_1}$
$N_1 + 2N_2 \geq M_1 + M_2 > N_1 + \frac{N_2}{2}$, $(N_1+N_2-M_1)^2 \geq (M_2+N_2)(N_1-M_1)$	$\frac{M_1(M_2+N_2)+M_2(N_1+N_2-M_1)}{N_2(M_2+N_2)}$	$\frac{2N_1+N_2}{2M_1}$
$M_1 + M_2 \geq N_1 + 2N_2$, $N_1 - M_1 > N_2$	$\frac{M_1(N_1+3N_2-M_1)+(N_1+2N_2-M_1)(N_1+N_2-M_1)}{N_2(N_1+N_2-M_1)}$	$\frac{2N_1+N_2}{2M_1}$
$M_1 + M_2 \geq N_1 + 2N_2$, $(N_1+N_2-M_1)^2 < (M_2+N_2)(N_1-M_1)$	$\frac{N_1(N_1-M_1)+N_2(N_1+N_2)}{N_2(N_1+N_2-M_1)}$	$\frac{2N_1+N_2}{2M_1}$

In Figure 2.3, we compare the number of DoF achieved by the proposed transmission scheme to the corrected number of DoF achieved by the transmission scheme of [KA17]. Additionally, the transmission scheme of [GAK12] is added to the comparison. In the region of antenna configurations of $\frac{3}{2} < \frac{M}{N} < 2$, the number of DoF achieved using the proposed transmission scheme is greater than that achieved by the schemes of [KA17] and [GAK12].

2.7 Conclusion

In this chapter, the MIMO XC with delayed CSIT was considered. For the transmission scheme given in [KA17], we performed decodability analysis of the transmitted information symbols. As a result of our analysis, we identified that the information symbols transmitted using [KA17] are not always decodable, which is due to a linear dependence of the linear combinations used for decoding at the receivers. To address the decodability problem, a novel transmission scheme was proposed, where the parameters of the transmission were chosen to maximize the number of the transmitted information symbols while ensuring linear independence of the linear combinations. The proposed transmission scheme achieves a number of DoF greater than that of [KA17] in which the number of the transmitted information symbols is reduced to the number of the decodable ones.

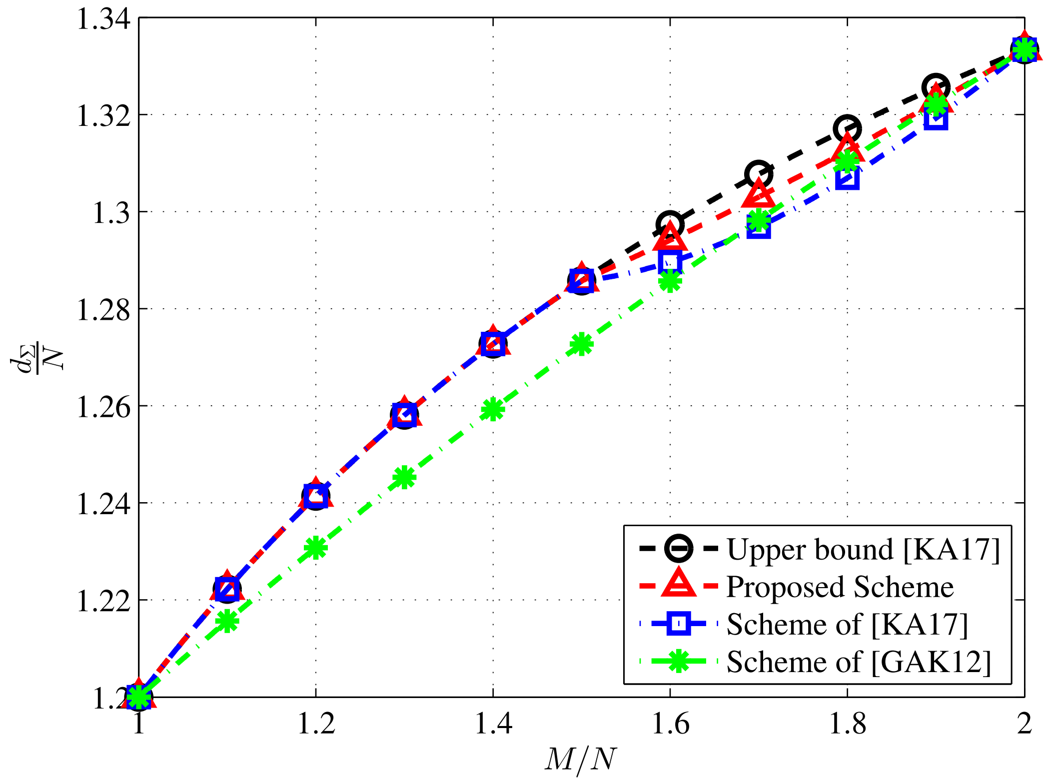

Figure 2.3: The number of DoF of the symmetric MIMO XC.

Chapter 3

Achievable DoF of the 3-User Symmetric MIMO Interference Channel with Delayed CSIT

3.1 Introduction

In this chapter, the three-user MIMO IC with delayed CSIT is considered. We focus on the symmetric MIMO setting, where each transmitter has M antennas and each receiver has N antennas. As we already mentioned in Section 1.1, to exploit the delayed in the three-user SISO IC, the RT-PIN transmission has to be applied. By applying RT-PIN, a number of DoF achievability results has been obtained in the literature: [MJS12] has achieved $\frac{9}{8}$ DoF and [AGK13] achieved $\frac{31}{36}$ DoF. As we already mentioned in Section 1.1, the DoF achievability schemes for the SISO case can be readily extended to the symmetric MIMO case by restricting each transmitter and each receiver to use $\min\{M, N\}$ antennas, where the DoF achieved in the SISO case scale by the factor $\min\{M, N\}$. In this case however, additional transmit and receive antennas are not exploited.

For $M < N$, as initially proposed in [GAK12], additional receive antennas can be exploited by taking into account the fact that the redundancy is naturally introduced by the channel, where the amount of the additionally introduced redundancy has to be adjusted according to the ratio $\frac{M}{N}$. Based on this approach, [TAV14] proposed for the three-user symmetric MIMO IC the transmission based on the scheme in [MJS12]. For $M > N$, as initially proposed in [KA17], the additional transmit antennas can be exploited by resorting to the IS-RT approach, which reduces the sizes of the signal spaces of individual transmitters at the unintended receiver while simultaneously ensuring that signals of individual transmitters span the receive signal space of the unintended receiver only partially. For the K-user symmetric MIMO XC, [HC16] proposed the transmission where in phase 1, IS-RT-PIN has been applied and the remaining phases relied on RT-PIN. To combat the linear dependence, [HC16] restricted the maximum number of used transmit antennas, and for large $\frac{M}{N}$, the MAT-like transmission has been employed by scheduling only two transmitters to transmit simultaneously. As for the optimality of the proposed transmission schemes, no optimality has been stated except for the trivial antenna configurations. The only non-trivial upper bound on the linear DoF has been proposed in [LAS14] for the SISO three-user IC.

The contributions of this chapter are the following. Firstly, we propose two novel transmission schemes for the three-user MIMO IC with delayed CSIT. The transmission schemes rely on the three-phase structure of the transmission scheme in [AGK13], where depending on whether $M < N$ or $M > N$, two different design approaches are applied. For $M < N$, the transmissions in phases 1 and 2 rely on the RT-PIN approach, where similarly to [GAK12] and [TAV14], we adjust the amount of the additionally introduced redundancy according to the ratio $\frac{M}{N}$. For $M > N$, the transmissions in phases 1 and 2 rely on the IS-RT-PIN approach. In phase 1, as compared to the two-part IS-RT applied in [HC16], a novel three-part IS-RT is employed. The IS and RT parts of the proposed IS-RT have different durations for different transmitters, which allows to reduce the number of linearly dependent linear combinations where the number of the transmit antennas is restricted only at a single transmitter. In phase 2, a more effective IS-RT is applied, as compared to the simpler RT in [HC16]. For both $M < N$ and $M > N$, the proposed transmission schemes achieve more DoF as compared to that achieved in the literature for a range of antenna configurations.

Secondly, an upper bound on the achievable DoF is proposed assuming linear coding strategies. The derivation of the proposed upper bound follows the footsteps of the derivation of the upper bound proposed for the three-user SISO IC in [LAS14]. The proof in [LAS14] relies on an upper bound on the ratio of the sizes of the signal spaces of two transmitters at intended and unintended receivers. We extend the proof of [LAS14] to a MIMO setting where for the upper bound on the ratio of the sizes of the signal spaces, we apply the upper bound proposed in [KA17]. Comparing the proposed upper bound with the DoF achieved in the literature shows that the proposed upper bound is tight for the antenna configurations of $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ and $2 \leq \frac{M}{N} < 3$. Parts of the results of this chapter has been published by the author of this thesis in [BASK15] and [BASK17].

The rest of the chapter is organized as follows. The system model will be given in Section 3.2. The main results and the comparison with the works existing in the literature will be gived in Section 3.3. The proposed transmission schemes will be given in Sections 3.4.2 and 3.4.3 for the cases $M < N$ and $M > N$, respectively. The conclusions of the chapter will be given in Section 3.5.

3.2 System Model

We consider a 3-user MIMO IC which is comprised of three transmitters Tx_i, $i \in \{1, 2, 3\}$ and three receivers Rx_j, $j \in \{1, 2, 3\}$, which is depicted in Figure 3.1. Each of the transmitters is equipped with M antennas and each of the receivers is equipped with N antennas. The signal received by Rx_j, $j \in \{1, 2, 3\}$, at the t-th channel use is evaluated as

$$\mathbf{y}_j(t) = \mathbf{H}_{j1}(t)\,\mathbf{x}_1(t) + \mathbf{H}_{j2}(t)\,\mathbf{x}_2(t) + \mathbf{H}_{j3}(t)\,\mathbf{x}_3(t) + \mathbf{z}_j(t), \quad (3.1)$$

where $\mathbf{x}_i(t)$ is the signal transmitted by Tx_i, $i \in \{1, 2, 3\}$, $\mathbf{H}_{ji}(t)$ is the channel matrix between Tx_i and Rx_j, and $\mathbf{z}_j(t) \sim \mathcal{CN}(0, \mathbf{I}_N)$ is the additive white Gaussian noise at Rx_j. The channel coefficients are drawn from a continuous distribution and are i.i.d. for different transmitter and receiver pairs as well as for different antennas and channel uses. The signals transmitted by Tx_i are subject to the transmit power constraint $\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}\left\{\mathbf{x}_i^{\mathrm{H}}(t)\,\mathbf{x}_i(t)\right\} \leq P$, where n is the communication duration.

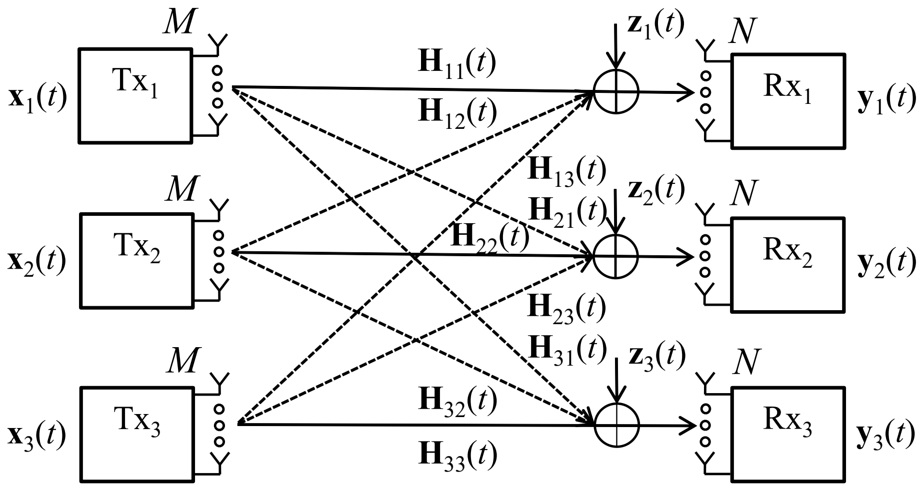

Figure 3.1: Three-user symmetric MIMO IC

Let us denote the set of all channel matrices for different transmitter and receiver pairs up to the t-th channel use as

$$\mathcal{H}^t = \{\mathbf{H}_{ji}(\tau) \mid i, j \in \{1, 2, 3\},\ \tau = 1, \ldots, t\}. \quad (3.2)$$

The delayed CSIT setting is introduced as the following knowledge about the channel matrices at the transmitter and receiver side.

- At the t-th channel use, $\mathcal{H}^t$ is known at every receiver.
- At the t-th channel use, $\mathcal{H}^{t-1}$ is known at every transmitter.

In this chapter, we restrict ourselves to linear coding strategies, for which the DoF correspond to the sum of dimensions of useful signal spaces of all transmitter to receiver pairs normalized with respect to the communication duration [BCT14]. The transmission is performed over the communication duration n during which Tx_i, $i \in \{1,2,3\}$, intends to deliver a vector $\mathbf{u}_i \in \mathbb{C}^{b_i(n)\times 1}$ of $b_i(n)$ information symbols to Rx_i. At the t-th channel use, Tx_i linearly precodes the information symbols as

$$\mathbf{x}_i(t) = \mathbf{C}_i(t)\,\mathbf{u}_i, \tag{3.3}$$

where $\mathbf{C}_i(t) \in \mathbb{C}^{M\times b_i(n)}$ is the precoding matrix. Let us denote the vertical concatenation of the signals transmitted by Tx_i up to the time instant n as $\mathbf{x}_i^n \in \mathbb{C}^{nM\times 1}$ and the vertical concatenation of the corresponding precoding matrices used by Tx_i as $\mathbf{C}_i^n \in \mathbb{C}^{nM\times b_i(n)}$, respectively. The vertical concatenation of the signals received by Rx_j is given by

$$\mathbf{y}_j^n = \mathbf{H}_{j1}^n\mathbf{C}_1^n\mathbf{u}_1 + \mathbf{H}_{j2}^n\mathbf{C}_2^n\mathbf{u}_2 + \mathbf{H}_{j3}^n\mathbf{C}_3^n\mathbf{u}_3 + \mathbf{z}_j^n \in \mathbb{C}^{nN\times 1}, \tag{3.4}$$

where $\mathbf{H}_{ji}^n \in \mathbb{C}^{nN\times nM}$ is a diagonal concatenation of the channel matrices between Tx_i and Rx_j, and $\mathbf{z}_j^n \in \mathbb{C}^{nN\times 1}$ is the vertical concatenation of the receive noise vectors.

Following the approach of [LAS14], we introduce the condition on decodability of $\mathbf{u}_i$. Let

$$\mathcal{I}_i = \mathrm{span}\left(\bigcup_{1\le i\le 3, i\ne j} \mathbf{H}_{ji}^n\mathbf{C}_i^n\right) \subseteq \mathbb{C}^{nN\times 1} \tag{3.5}$$

denote the subspace at Rx_j containing the interference signal, where $\mathrm{span}(\mathbf{A}) = \left\{\mathbf{A}\mathbf{x}\,\middle|\,\mathbf{x}\in\mathbb{C}^{k\times 1}\right\}$, is a space spanned by the columns of $\mathbf{A}\in\mathbb{C}^{l\times k}$. The interference-free subspace is then given by $\mathcal{I}_i^{\mathrm{c}} \subseteq \mathbb{C}^{nN\times 1}$, where $\mathcal{A}^{\mathrm{c}} \subseteq \mathbb{C}^{l\times 1}$ denotes the subspace complementary to $\mathcal{A}\subseteq\mathbb{C}^{l\times 1}$. Given two subspaces $\mathcal{A},\mathcal{B}\subseteq\mathbb{C}^{l\times 1}$, the projection of $\mathcal{B}$ on $\mathcal{A}$ is given by $\mathrm{proj}_{\mathcal{A}}\mathcal{B} = \left\{\mathbf{x}\in\mathcal{A}\,\middle|\,\exists\mathbf{y}\in\mathcal{B},\ \text{s.t. } \mathbf{x}^{\mathrm{H}}\mathbf{y}\ne 0\right\} \subseteq \mathbb{C}^{l\times 1}$. In the limit of large P where the noise can be neglected, the vector of information symbols $\mathbf{u}_i$ is said to be decodable if

$$\dim\left(\mathrm{proj}_{\mathcal{I}_i^{\mathrm{c}}}\mathrm{span}\left(\mathbf{H}_{ii}^n\mathbf{C}_i^n\right)\right) = b_i(n), \tag{3.6}$$

where $\dim(.)$ gives the dimension of the subspace.

Further following [LAS14], we give the definition of the number of DoF as follows. The DoF tuple (d_1,d_2,d_3) is deemed to be linearly achievable if there exists a sequence in n of precoding matrix tuples $(\mathbf{C}_1^n,\mathbf{C}_2^n,\mathbf{C}_3^n)$ fulfilling the decodability constraint (3.6) for which

$$d_i = \lim_{n\to\infty}\frac{b_i(n)}{n} \tag{3.7}$$

holds. The closure of all linearly achievable DoF tuples (d_1,d_2,d_3) is called the DoF region $\mathcal{D}_{\mathrm{lin}}$. The linearly achievable sum-DoF is denoted by $d_\Sigma = d_1+d_2+d_3$. The maximum linearly achievable sum-DoF (or simply linear DoF) is given by

$$d_{\mathrm{lin}} = \max_{(d_1,d_2,d_3)\in\mathcal{D}_{\mathrm{lin}}} d_1+d_2+d_3. \tag{3.8}$$

3.3 Main Results

In the this section, we state the main results of the chapter. We start with a theorem stating the DoF achievability.

Theorem 1. *For the 3-user MIMO IC with delayed CSIT, the following sum-DoF is achievable:*

$$d_{\Sigma} = \begin{cases} \frac{9MN}{3N+4M}, & \frac{3}{4} < \frac{M}{N} \leq \frac{4}{5}, \\ \frac{36N}{31}, & \frac{4}{5} < \frac{M}{N} < 1, \\ \frac{36MN}{17M+14N}, & 1 < \frac{M}{N} \leq \frac{3}{2}, \\ \frac{12MN(6N-M)}{-7M^2+34MN+24N^2}, & \frac{3}{2} < \frac{M}{N} \leq \frac{5}{3}, \\ \frac{12N(6N-M)(6M^2-15MN+10N^2)}{-42M^3+321M^2N-552MN^2+284N^3}, & \frac{5}{3} < \frac{M}{N} < 2. \end{cases} \tag{3.9}$$

The proof of Theorem 1 will be given in Section 3.4 by describing the corresponding DoF achievability schemes.

In addition to the results on the DoF achievability, we give an upper bound on the linear DoF in the following theorem.

Theorem 2. *For the 3-user MIMO IC with delayed CSIT, the linear DoF are upper bounded as*

$$d_{lin} \leq \begin{cases} \frac{3MN}{M+N}, & \frac{1}{2} < \frac{M}{N} \leq \frac{3}{4}, \\ \frac{9N}{7}, & \frac{3}{4} < \frac{M}{N} \leq 1, \\ \frac{9MN}{5M+2N}, & 1 < \frac{M}{N} \leq 2, \\ \frac{3N}{2}, & 2 < \frac{M}{N} < 3. \end{cases} \tag{3.10}$$

The proof of Theorem 2 is relegated to section A.1.

The upper bound on the linear DoF given in Theorem 2 was previously achieved by the work of [TAV14] for the region of $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$, and by the work of [HC15] for the region of $2 \leq \frac{M}{N} < 3$, hence the following corollary follows.

Corollary 1. *For the 3-user MIMO IC with delayed CSIT, the linear DoF are*

$$d_{lin} = \begin{cases} \frac{3MN}{M+N}, & \frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}, \\ \frac{3N}{2}, & 2 \leq \frac{M}{N} < 3. \end{cases} \tag{3.11}$$

The achievable DoF proposed in Theorem 1 as well as the upper bound on the linear DoF proposed in Theorem 2 are plotted for the regions of the antenna configurations of $\frac{1}{2} \leq \frac{M}{N} \leq 1$ and $1 \leq \frac{M}{N} \leq 2$ in Figures 3.2 and 3.3, respectively. In addition to that, we depict the DoF achieved in the existing works [TAV14] and [HC16] as well as the DoF achieved by the works [TAV16] and [CSG16] which appeared in the literature in parallel or after the publication of the results given in this chapter.

1. $\frac{1}{2} < \frac{M}{N} < 1$ (Fig. 3.2):

    - *Comparison with [TAV14]:* The achievable DoF proposed by Theorem 1 are greater for $\frac{3}{4} < \frac{M}{N} < 1$.
    - *Comparison with [TAV16]:* The achievable DoF proposed by Theorem 1 are greater for $\frac{3}{4} < \frac{M}{N} < \frac{4}{5}$, where for $\frac{4}{5} < \frac{M}{N} < 1$, [TAV16] achieves the DoF identical to that proposed by Theorem 1.
    - *Comparison with [CSG16]:* The achievable DoF proposed by Theorem 1 are greater for $\frac{3}{4} < \frac{M}{N} < \frac{12}{13}$, where for $\frac{12}{13} < \frac{M}{N} < 1$, [CSG16] achieves the DoF greater than that proposed by Theorem 1.

2. $1 < \frac{M}{N} < 2$ (Fig. 3.3):

    - *Comparison with [TAV16]:* The achievable DoF proposed by Theorem 1 are greater for the whole region $1 < \frac{M}{N} < 2$.
    - *Comparison with [HC16]:* The achievable DoF proposed by Theorem 1 are greater for the whole region $1 < \frac{M}{N} < 2$. In the region $1 < \frac{M}{N} < \frac{5}{3}$, the gain in the achievable DoF is due to a more effective transmission in phase 2, where the transmission in phase 1 is identical to that of [HC16]. In $\frac{5}{3} < \frac{M}{N} < 2$, both phases 1 and 2 have a more effective transmission which results in a greater gain in the achievable DoF.
    - *Comparison with [CSG16]:* The achievable DoF proposed by Theorem 1 are greater for the region of $\frac{14}{13} < \frac{M}{N} < 2$, where for $1 < \frac{M}{N} < \frac{14}{13}$, [CSG16] achieves the DoF greater than that proposed by Theorem 1.

With the exception of the regions $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ and $2 \leq \frac{M}{N} < 3$ mentioned in Corollary 1, in the region of $\frac{3}{5} < \frac{M}{N} < 2$ the upper bound on the linear DoF stated in Theorem 2 in not achieved by any of the DoF achievability schemes plotted in Figures 3.2 and 3.3, thus the DoF characterization for this region of the antenna configurations remains an open question.

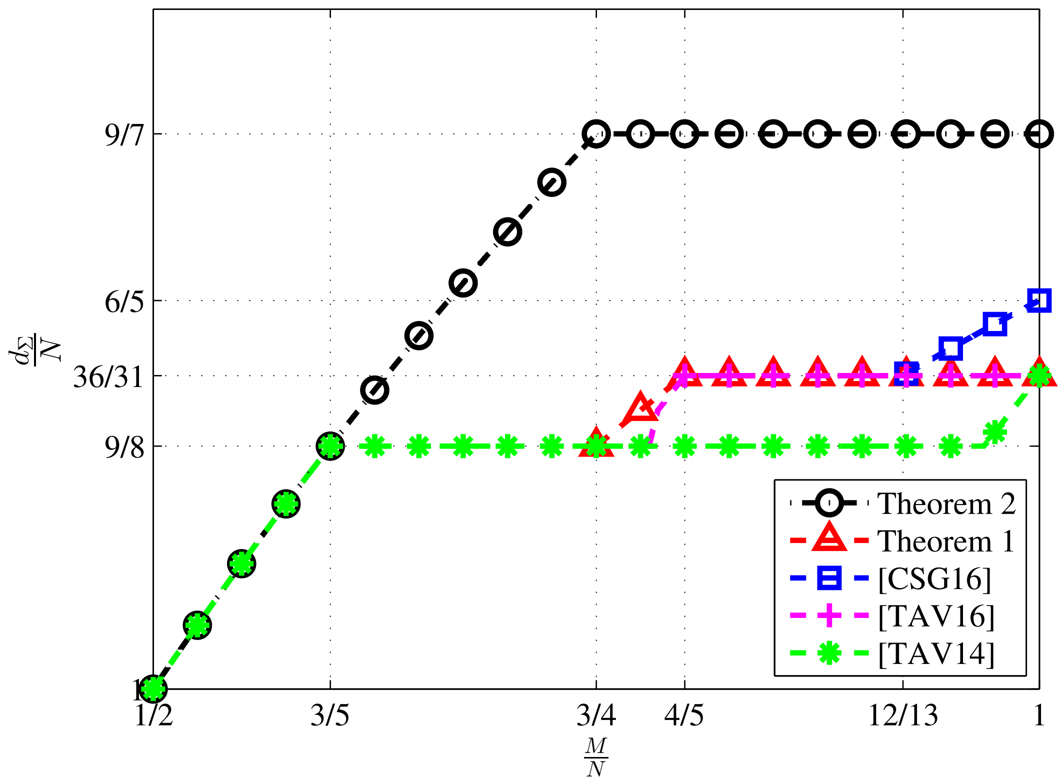

Figure 3.2: DoF of 3-user MIMO IC with delayed CSIT for $\frac{1}{2} \leq \frac{M}{N} \leq 1$

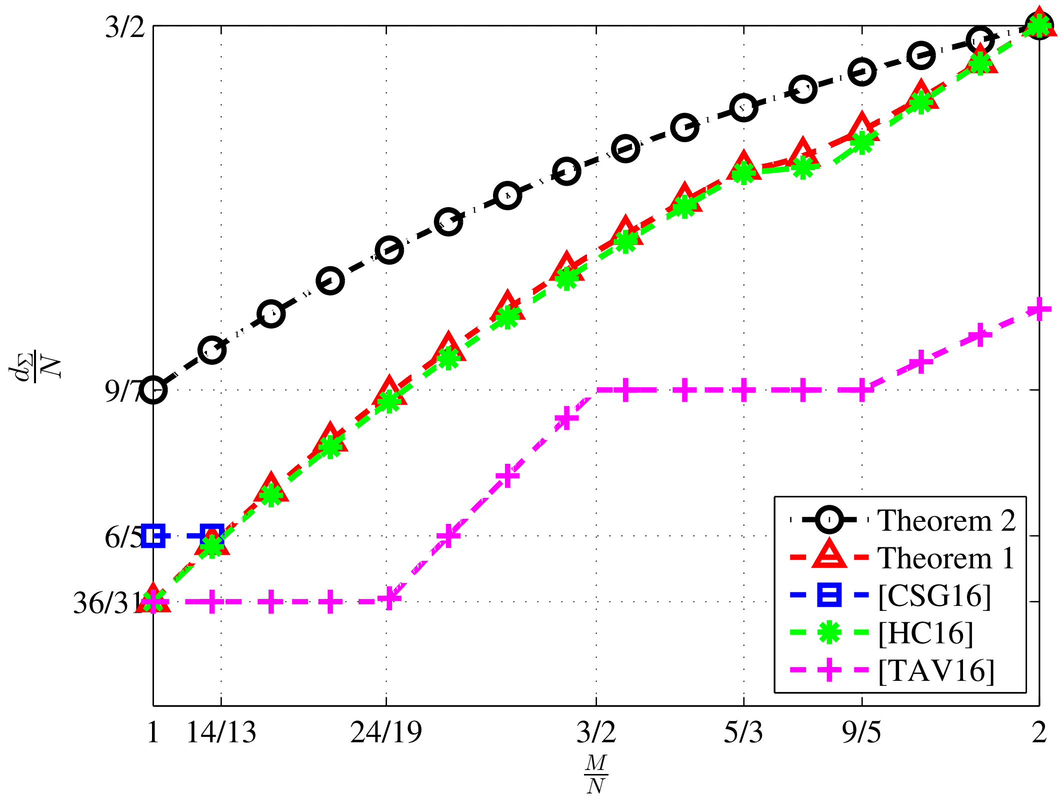

Figure 3.3: DoF of 3-user MIMO IC with delayed CSIT for $1 \leq \frac{M}{N} \leq 2$

3.4 Proof of Theorem 1

In this section, we prove Theorem 1 by describing the corresponding DoF achievability schemes.

As we already mentioned in Section 3.1, the propsed transmission scheme follows the three-phase structure of the transmission scheme in [AGK13]. Below we shortly summarize the details of the transmission.

Phase 1: In phase 1, all transmitters are scheduled to transmit simultaneously the original information symbols. Depending on whether $M < N$ or $M > N$, the RT or IS-RT approach is applied, which allows each receiver to cancel the signal of one of the interferers using PIN. The residual interference terms obtained after PIN comprise the order-2 symbols to be transmitted in phase 2.

Phase 2: In phase 2, the order-2 symbols generated in phase 1 are transmitted, where the transmitters are scheduled in pairs to transmit the order-2 symbols useful to the same pair of receivers. Similarly to phase 1, depending on whether $\frac{M}{N} \leq 1$ or $\frac{M}{N} > 1$, either RT or IS-RT is applied, which allows the third unintended receiver to obtain the terms comprised of the signals of only a single transmitter from the received signal. The interference terms obtained after PIN comprise the terms useful for two receivers and known at the remaining third receiver, referred to in the following as order-(2,1) symbols. The delivery of the order-(2,1) symbols is performed in phase 3.

Phase 3: In phase 3, all transmitters simultaneously transmit the order-(2,1) symbols generated in phase 2. Each of the receivers uses the known order-(2,1) symbols obtained in phase 2 to cancel the interference in the received signal, thus interference-free reception is achieved. With the delivery of the order-(2,1) symbols in phase 3, the transmission is finished and the receivers proceed with the decoding of the desired information symbols.

3.4.1 Transmission Blocks

In the following section, we introduce the notion of the transmission blocks as given in Figure 3.4, which are transmission periods with identical structure, but different transmitted information symbols. Splitting each phase into transmission blocks simplifies the design of the transmission scheme, where for each phase the structure of the transmission blocks can be designed independently.

For a transmission block of phase $l \in \{1,2,3\}$ having a duration of $T^{(l)}$ time slots, each of the active transmitters transmits $b_i^{(l)}$ symbols with $b_\Sigma^{(l)}$ denoting the number of the information symbols transmitted by all transmitters. After the transmission of a transmission block of phase $l \in \{1,2\}$, $q_\Sigma^{(l)}$ terms to be transmitted in phase $l+1$ are generated. The number of the transmission blocks of phase $l \in \{1,2,3\}$ is denoted as $k^{(l)}$.

After the design of the transmission blocks, the number of the transmission blocks of each phase is carefully selected such that all generated higher-order symbols are delivered to the receivers which desire them. The delivery of all generated higher-order symbols is possible, if the number $k^{(l)}q_\Sigma^{(l)}$ of the terms generated in phase $l \in \{1,2\}$ equals the number $k^{(l+1)}b_\Sigma^{(l+1)}$ of the information symbols transmitted in phase $l+1$:

$$k^{(l)}q_\Sigma^{(l)} = k^{(l+1)}b_\Sigma^{(l+1)}, l \in \{1,2\}\,. \tag{3.12}$$

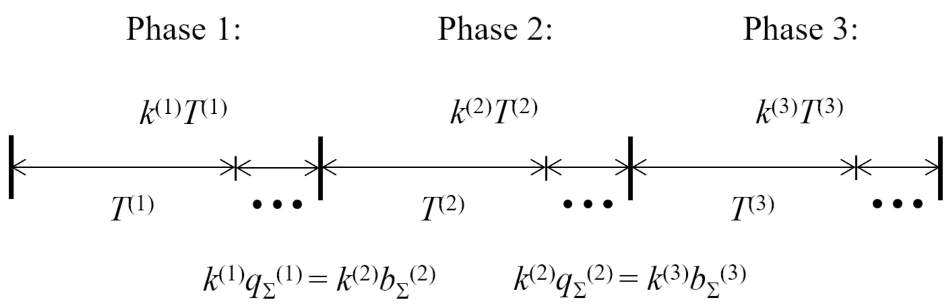

Figure 3.4: Transmission blocks

To describe the transmission scheme, we will first give the structure of the transmission block of each phase. In the end, the number of the transmission blocks of each phase will be calculated.

3.4.2 Case of $M < N$

In this section, we prove Theorem 1 for $M < N$ by describing the DoF achievability scheme employing the RT-PIN concept. First, the transmission blocks of each phase will be introduced. Then, the numbers of the transmission blocks and the achieved DoF will be calculated.

3.4.2.1 Phase 1

In phase 1, the original information symbols are transmitted. In a single transmission block of phase 1 of duration $T^{(1)}$, all transmitters are scheduled to transmit simultaneously, where Tx_i, $i \in \{1,2,3\}$, transmits $b_i^{(1)} = b^{(1)}$ information symbols comprising the information symbol vector $\mathbf{u}_i^{(1)} \in \mathbb{C}^{b^{(1)} \times 1}$.

RT: For the transmission of the information symbols, each of the transmitters applies redundant precoding where the number $b^{(1)}$ of the transmitted information symbols is restricted as $\frac{NT^{(1)}}{2} \leq b^{(1)} \leq NT^{(1)}$. The vertical concatenation of the signals transmitted by Tx_i in the transmission block is evaluated as

$$\mathbf{x}_i^{(1)} = \mathbf{C}_i^{(1)} \mathbf{u}_i^{(1)} \in \mathbb{C}^{MT^{(1)} \times 1}, \tag{3.13}$$

where $\mathbf{C}_i^{(1)} \in \mathbb{C}^{MT^{(1)} \times b^{(1)}}$ is the random matrix with i.i.d. continuously distributed precoding coefficients. By omitting the receive noise, the signal received by Rx_j, $j \in \{1,2,3\}$, is evaluated as

$$\mathbf{y}_j^{(1)} = \mathbf{H}_{j1}^{(1)} \mathbf{C}_1^{(1)} \mathbf{u}_1^{(1)} + \mathbf{H}_{j2}^{(1)} \mathbf{C}_2^{(1)} \mathbf{u}_2^{(1)} + \mathbf{H}_{j3}^{(1)} \mathbf{C}_3^{(1)} \mathbf{u}_3^{(1)}, \tag{3.14}$$

where $\mathbf{H}_{ji}^{(1)} \in \mathbb{C}^{NT^{(1)} \times MT^{(1)}}$ is the channel matrix between Tx_i and Rx_j and $\mathbf{n}_j^{(1)} \in \mathbb{C}^{NT^{(1)} \times 1}$ is the vertical concatenation of the noise vectors at Rx_j.

PIN: For the generation of the overheard interference terms, each of the receivers has to cancel the signal of one of the interferers. We describe PIN by giving the processing at Rx_1, where the processing at other receivers is performed similarly.

The signal $\mathbf{y}_1^{(1)}$ received by Rx_1 is comprised of the useful terms $\mathbf{H}_{11}^{(1)} \mathbf{C}_1^{(1)} \mathbf{u}_1^{(1)}$ and two interference terms $\mathbf{H}_{12}^{(1)} \mathbf{C}_2^{(1)} \mathbf{u}_2^{(1)}$ and $\mathbf{H}_{13}^{(1)} \mathbf{C}_3^{(1)} \mathbf{u}_3^{(1)}$, useful for Rx_2 and Rx_3, respectively. First, let us consider the interference nulling of the interference term $\mathbf{H}_{12}^{(1)} \mathbf{C}_2^{(1)} \mathbf{u}_2^{(1)}$. Since both the channel matrix $\mathbf{H}_{12}^{(1)}$ and the precoding matrix $\mathbf{C}_2^{(1)}$ are distributed independently, the matrix product $\mathbf{H}_{12}^{(1)} \mathbf{C}_2^{(1)}$ is almost surely full rank with $\left(NT^{(1)} - b^{(1)}\right)$ dimensional left null space. Let us denote by $\mathbf{W}_{12}^{(1)} \in \mathbb{C}^{\left(NT^{(1)} - b^{(1)}\right) \times NT^{(1)}}$ the matrix whose columns form a basis in the left null space of $\mathbf{H}_{12}^{(1)} \mathbf{C}_2^{(1)}$. By a projection

$$\mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{y}_1^{(1)} = \mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{H}_{11}^{(1)} \mathbf{C}_1^{(1)} \mathbf{u}_1^{(1)} + \mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{H}_{13}^{(1)} \mathbf{C}_3^{(1)} \mathbf{u}_3^{(1)} \tag{3.15}$$

the interference term $\mathbf{H}_{12}^{(1)} \mathbf{C}_2^{(1)} \mathbf{u}_2^{(1)}$ is cancelled from the received signal. The remaining interference term $\mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{H}_{13}^{(1)} \mathbf{C}_3^{(1)} \mathbf{u}_3^{(1)}$ is useful for both Rx_1 and Rx_3 as follows.

- By subtracting $\mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{y}_1^{(1)} - \mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{H}_{13}^{(1)} \mathbf{C}_3^{(1)} \mathbf{u}_3^{(1)} = \mathbf{W}_{12}^{(1)\mathrm{H}} \mathbf{H}_{11}^{(1)} \mathbf{C}_1^{(1)} \mathbf{u}_1^{(1)}$, a term useful for Rx_1 is obtained.

- It is a term useful for Rx_3.

By setting $\mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{H}_{13}^{(1)}\mathbf{C}_3^{(1)}\mathbf{u}_3^{(1)} = \mathbf{u}_{3|1,3} \in \mathbb{C}^{\left(NT^{(1)}-b^{(1)}\right)\times 1}$, a vector of order-2 symbols is generated, where $\mathbf{u}_{l|i,j}$ is a vector of order-2 symbols simultaneously useful for Rx_i and Rx_j, which is comprised of the information symbols available at Tx_l, $1 \leq i < j \leq 3$, $l \in \{i, j\}$.

Similarly, the interference term $\mathbf{H}_{13}^{(1)}\mathbf{C}_3^{(1)}\mathbf{u}_3^{(1)}$ can be cancelled from $\mathbf{y}_1^{(1)}$ at Rx_1 by projecting the received signal on the columns of the matrix $\mathbf{W}_{13}^{(1)} \in \mathbb{C}^{\left(NT^{(1)}-b^{(1)}\right)\times NT^{(1)}}$, which form a basis in the left null space of $\mathbf{H}_{13}^{(1)}\mathbf{C}_3^{(1)}\mathbf{u}_3^{(1)}$. After the projection, the vector of order-2 symbols $\mathbf{u}_{2|1,2} = \mathbf{W}_{13}^{(1)\mathrm{H}}\mathbf{H}_{12}^{(1)}\mathbf{C}_2^{(1)} \in \mathbb{C}^{\left(NT^{(1)}-b^{(1)}\right)\times 1}$ will be generated. By following the same steps, order-2 symbols $\mathbf{u}_{3|2,3}, \mathbf{u}_{1|1,2} \in \mathbb{C}^{\left(NT^{(1)}-b^{(1)}\right)\times 1}$ will be generated from the signal $\mathbf{y}_2^{(1)}$ at Rx_2 and the order-2 symbols $\mathbf{u}_{2|2,3}, \mathbf{u}_{1|1,3} \in \mathbb{C}^{\left(NT^{(1)}-b^{(1)}\right)\times 1}$ will be generated from the signal $\mathbf{y}_3^{(1)}$ at Rx_3, where the details are omitted to avoid repetition. In total, $6\left(NT^{(1)} - b^{(1)}\right)$ order-2 symbols are generated using PIN at all receivers. The generated order-2 symbols will be reconstructed at the corresponding transmitters using the delayed CSIT and saved for the transmission in phase 2.

Choice of $b^{(1)}$ and $T^{(1)}$: To maximize the achievable DoF, the parameters $b^{(1)}$ and $T^{(1)}$ are designed to maximize the normalized number $\frac{b^{(1)}}{T^{(1)}}$ of the information symbols while ensuring that the information symbols can be decoded given the generated order-2 symbols are delivered to the receivers which desire them. In the following, two decodability bounds $B_1^{(1)}$ and $B_2^{(1)}$ are introduced:

- The number $b^{(1)}$ of the information symbols transmitted to Rx_i, $i \in \{1, 2, 3\}$, has to be less than or equal to the number $4\left(NT^{(1)} - b^{(1)}\right)$ of the linear combinations provided to Rx_i by the order-2 symbols, which is rewritten as a bound

$$B_1^{(1)} \equiv \frac{b^{(1)}}{T^{(1)}} \leq \frac{4N}{5}. \tag{3.16}$$

- The number $b^{(1)}$ of the information symbols transmitted by Tx_i has to be less than or equal to the total number $MT^{(1)}$ of the transmit dimensions, which is rewritten as a bound

$$B_2^{(1)} \equiv \frac{b^{(1)}}{T^{(1)}} \leq M. \tag{3.17}$$

Depending on whether $B_1^{(1)}$ or $B_2^{(1)}$ are active, two regions of antenna configurations are distinguished. The choice of $b^{(1)}$ and $T^{(1)}$ for them is given below.

1. $\frac{3}{4} < \frac{M}{N} \leq \frac{4}{5}$ ($B_2^{(1)}$ is active): In order to maximize $\frac{b^{(1)}}{T^{(1)}}$ while ensuring $B_2^{(1)}$ holds, we choose

$$b^{(1)} = 4M, \; T^{(1)} = 4. \tag{3.18}$$

Since $B_1^{(1)}$ is inactive, the number of the generated order-2 symbols exceeds the number of the order-2 symbols necessary for the decoding. To avoid unnecessary retransmissions, we take only the first $\frac{MT^{(1)}}{4} = M$ elements of each of the generated order-2 symbol vectors for the transmission in phase 2, where the total number of the order-2 symbols chosen for the retransmission is given by

$$q_{\Sigma}^{(1)} = 6M. \tag{3.19}$$

2. $\frac{4}{5} < \frac{M}{N} < 1$ ($B_1^{(1)}$ is active): In order to maximize $\frac{b^{(1)}}{T^{(1)}}$ while ensuring $B_1^{(1)}$ holds, we choose

$$b^{(1)} = 4N, \; T^{(1)} = 5. \tag{3.20}$$

Since $B_1^{(1)}$ is active, all $\left(NT^{(1)} - b^{(1)}\right) = N$ elements of each of the generated order-2 symbol vectors are chosen for the transmission in phase 2, where the total number of the order-2 symbols to be retransmitted in phase 2 is given by

$$q_{\Sigma}^{(1)} = 6N. \tag{3.21}$$

The order-2 symbols chosen for the retransmission will be reconstructed at the corresponding transmitters using the delayed CSIT, and transmitted in phase 2, which we describe in the following.

3.4.2.2 Phase 2

In phase 2, the order-2 symbols generated in phase 2 are transmitted, where the transmitters are scheduled to transmit in pairs. For a given scheduled transmitter pair $(\text{Tx}_i, \text{Tx}_j)$, $i, j \in \{1,2,3\}$, $i \neq j$, each of the transmitters transmits $b_i^{(2)} = b_j^{(2)} = b^{(2)}$ order-2 symbols. The order-2 symbols transmitted by Tx_i and Tx_j are useful for the same pair of receivers Rx_i and Rx_j, where the order-2 symbol vectors transmitted by Tx_i and Tx_j are given by $\mathbf{u}_{i|i,j}^{(2)} \in \mathbb{C}^{b^{(2)} \times 1}$ and $\mathbf{u}_{j|i,j}^{(2)} \in \mathbb{C}^{b^{(2)} \times 1}$, respectively. To avoid repetition, we describe only the case where Tx_1 and Tx_2 transmit the order-2 symbol vectors $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$ to Rx_1 and Rx_2, respectively. The transmissions for the remaining receiver pairs can be obtained by changing the corresponding indices.

RT: Similarly to phase 1, for the transmission of the order-2 symbols Tx_1 and Tx_2 apply redundant precoding, where the number $b^{(2)}$ of the transmitted order-2 symbols is restricted as $\frac{T^{(2)}N}{2} \leq b^{(2)} \leq T^{(2)}N$. The vertical concatenations of the signals transmitted by Tx_1 and Tx_2 in the transmission block are given by

$$\mathbf{x}_1^{(2)} = \mathbf{C}_{1|1,2}^{(2)}\mathbf{u}_{1|1,2}^{(2)} \in \mathbb{C}^{MT^{(2)}\times 1},$$
$$\mathbf{x}_2^{(2)} = \mathbf{C}_{2|1,2}^{(2)}\mathbf{u}_{2|1,2}^{(2)} \in \mathbb{C}^{MT^{(2)}\times 1}, \tag{3.22}$$

respectively, where $\mathbf{C}_{1|1,2}^{(2)}, \mathbf{C}_{2|1,2}^{(2)} \in \mathbb{C}^{MT^{(2)}\times b^{(2)}}$ are the random matrices with i.i.d. continuously distributed precoding coefficients. By omitting the receive noise, the signal received by Rx_j, $j \in \{1,2,3\}$, is evaluated as

$$\mathbf{y}_j^{(1)} = \mathbf{H}_{j1}^{(2)}\mathbf{C}_{1|1,2}^{(2)}\mathbf{u}_{1|1,2}^{(2)} + \mathbf{H}_{j2}^{(2)}\mathbf{C}_{2|1,2}^{(2)}\mathbf{u}_{2|1,2}^{(2)}, \tag{3.23}$$

where $\mathbf{H}_{ji}^{(2)}$ is the channel matrix between Tx_i and Rx_j, $i \in \{1,2\}$.

PIN: For the generation of the overheard interference terms, Rx_3 has to alternatingly cancel the signals of Tx_1 and Tx_2 to obtain two terms containing the signals of only a single transmitter. The interference terms comprising the signal received by Rx_3 are given by $\mathbf{H}_{31}^{(1)}\mathbf{C}_{1|1,2}^{(2)}\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{H}_{32}^{(1)}\mathbf{C}_{2|1,2}^{(2)}\mathbf{u}_{2|1,2}^{(2)}$. Let us denote by $\mathbf{W}_{31}^{(2)} \in \mathbb{C}^{\left(NT^{(2)}-b^{(2)}\right)\times NT^{(2)}}$ and $\mathbf{W}_{32}^{(2)} \in \mathbb{C}^{\left(NT^{(2)}-b^{(2)}\right)\times NT^{(2)}}$ the projection matrices, the columns of which form bases in the left null spaces of $\mathbf{H}_{31}^{(2)}\mathbf{C}_{1|1,2}^{(2)}$ and $\mathbf{H}_{32}^{(2)}\mathbf{C}_{2|1,2}^{(2)}$, respectively. By the projections

$$\mathbf{W}_{31}^{(2)\mathrm{H}}\mathbf{y}_3^{(2)} = \mathbf{W}_{31}^{(2)\mathrm{H}}\mathbf{H}_{32}^{(2)}\mathbf{C}_{2|1,2}^{(2)}\mathbf{u}_{2|1,2}^{(2)} = \mathbf{u}_{2|1,2;3} \in \mathbb{C}^{\left(NT^{(2)}-b^{(2)}\right)\times 1}, \tag{3.24}$$
$$\mathbf{W}_{32}^{(2)\mathrm{H}}\mathbf{y}_3^{(2)} = \mathbf{W}_{32}^{(2)\mathrm{H}}\mathbf{H}_{31}^{(2)}\mathbf{C}_{1|1,2}^{(2)}\mathbf{u}_{1|1,2}^{(2)} = \mathbf{u}_{1|1,2;3} \in \mathbb{C}^{\left(NT^{(2)}-b^{(2)}\right)\times 1} \tag{3.25}$$

the terms useful for both Tx_1 and Tx_2 and available at Rx_3 are obtained, where $\mathbf{u}_{l|i,j;k}$ denotes an order-(2,1) symbol comprised of the symbols available at Rx_i, which is useful for Rx_i and Rx_j and is known at Rx_k, $i,j,k \in \{1,2,3\}$, $i \neq j \neq k$, $l \in \{i,j\}$.

Choice of $b^{(2)}$ and $T^{(2)}$: $b^{(2)}$ and $T^{(2)}$ are chosen to maximize $\frac{b^{(2)}}{T^{(2)}}$ while ensuring the transmitted order-2 symbols are decodable given all order-(2,1) symbols are delivered to the receivers which desire them. Similarly to phase 1, we introduce the following two decodability bounds $B_1^{(2)}$ and $B_2^{(2)}$.

- The number $2\left(NT^{(2)} - b^{(2)}\right)$ of the linear combinations which will provided to Rx_1 and Rx_2 by the order-(2,1) symbols has to be greater than or equal to the number $\left(2b^{(2)} - NT^{(2)}\right)$ of the missing linear combinations for decoding $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$, which is rewritten as a bound

$$B_1^{(2)} \equiv \frac{b^{(2)}}{T^{(2)}} \leq \frac{3N}{4}. \tag{3.26}$$

- The number $b^{(2)}$ of the order-2 symbols transmitted by Tx_i, $i \in \{1,2\}$, has to be less than or equal to the total number $MT^{(2)}$ of the transmit dimensions, which is rewritten as a bound
$$B_2^{(2)} \equiv \frac{b^{(2)}}{T^{(2)}} \leq M. \tag{3.27}$$

Since Theorem 1 assumes $\frac{3}{4} < \frac{M}{N}$, $B_1^{(2)}$ is always active. In order to maximize $\frac{b^{(2)}}{T^{(2)}}$ while ensuring $B_1^{(2)}$ holds, $b^{(2)}$ and $T^{(2)}$ are chosen as

$$b^{(2)} = 3N,\ T^{(2)} = 4, \tag{3.28}$$

where the total number of the generated order-(2,1) symbols is given by

$$q_{\Sigma}^{(2)} = 2\left(NT^{(2)} - b^{(2)}\right) = 2N. \tag{3.29}$$

Following the same methodology, the transmitter pairs $(\mathrm{Tx}_1, \mathrm{Tx}_3)$ and $(\mathrm{Rx}_2, \mathrm{Rx}_3)$ will be scheduled for the transmission, where the order-2 symbol vectors $\mathbf{u}_{3|1,3;2}, \mathbf{u}_{1|1,3;2} \in \mathbb{C}^{N\times 1}$ will be generated at Rx_2 and the order-2 symbol vectors $\mathbf{u}_{2|2,3;1}, \mathbf{u}_{3|2,3;1} \in \mathbb{C}^{N\times 1}$ will be generated at Rx_1. The order-(2,1) symbol vectors generated in phase 2 will be reconstructed at the corresponding transmitters using the delayed CSIT, and transmitted in phase 3.

3.4.2.3 Phase 3

In phase 3, the order-(2,1) symbol vectors generated in phase 2 are transmitted, where all transmitters are scheduled to transmit simultaneously. In a single transmission block, Tx_i, $i \in \{1,2,3\}$, transmits $b_i^{(3)} = b^{(3)}$ order-(2,1) symbols comprising two $\left(\frac{b^{(3)}}{2}\right)$-element order-(2,1) symbol vectors which are given below:

- Tx_1: $\mathbf{u}_{1|1,2;3}^{(3)}, \mathbf{u}_{1|1,3;2}^{(3)} \in \mathbb{C}^{\frac{b^{(3)}}{2}\times 1}$,
- Tx_2: $\mathbf{u}_{2|1,2;3}^{(3)}, \mathbf{u}_{2|2,3;1}^{(3)} \in \mathbb{C}^{\frac{b^{(3)}}{2}\times 1}$,
- Tx_3: $\mathbf{u}_{3|1,3;2}^{(3)}, \mathbf{u}_{3|2,3;1}^{(3)} \in \mathbb{C}^{\frac{b^{(3)}}{2}\times 1}$.

The transmitted signals are given by

$$\begin{aligned}
\mathbf{x}_1^{(3)} &= \mathbf{C}_{1|1,2;3}^{(3)}\mathbf{u}_{1|1,2;3}^{(3)} + \mathbf{C}_{1|1,3;2}^{(3)}\mathbf{u}_{1|1,3;2}^{(3)} \in \mathbb{C}^{MT^{(3)}\times 1},\\
\mathbf{x}_2^{(3)} &= \mathbf{C}_{2|1,2;3}^{(3)}\mathbf{u}_{2|1,2;3}^{(3)} + \mathbf{C}_{2|2,3;1}^{(3)}\mathbf{u}_{2|2,3;1}^{(3)} \in \mathbb{C}^{MT^{(3)}\times 1},\\
\mathbf{x}_3^{(3)} &= \mathbf{C}_{3|1,3;2}^{(3)}\mathbf{u}_{3|1,3;2}^{(3)} + \mathbf{C}_{3|2,3;1}^{(3)}\mathbf{u}_{3|2,3;1}^{(3)} \in \mathbb{C}^{MT^{(3)}\times 1},
\end{aligned} \tag{3.30}$$

where $\mathbf{C}_{1|1,2;3}^{(3)}, \mathbf{C}_{1|1,3;2}^{(3)}, \mathbf{C}_{2|1,2;3}^{(3)}, \mathbf{C}_{2|2,3;1}^{(3)}, \mathbf{C}_{3|1,3;2}^{(3)}, \mathbf{C}_{3|2,3;1}^{(3)} \in \mathbb{C}^{MT^{(3)}\times\frac{b^{(3)}}{2}}$ are the matrices with i.i.d. random precoding coefficients.

By omitting the receive noise, the signal received by Rx_j, $j \in \{1,2,3\}$, is evaluated as

$$\begin{aligned}
\mathbf{y}_j^{(3)} = \; & \mathbf{H}_{j1}^{(3)}\left(\mathbf{C}_{1|1,2;3}^{(3)}\mathbf{u}_{1|1,2;3}^{(3)} + \mathbf{C}_{1|1,3;2}^{(3)}\mathbf{u}_{1|1,3;2}^{(3)}\right) +\\
& \mathbf{H}_{j2}^{(3)}\left(\mathbf{C}_{2|1,2;3}^{(3)}\mathbf{u}_{2|1,2;3}^{(3)} + \mathbf{C}_{2|2,3;1}^{(3)}\mathbf{u}_{2|2,3;1}^{(3)}\right) +\\
& \mathbf{H}_{j3}^{(3)}\left(\mathbf{C}_{3|1,3;2}^{(3)}\mathbf{u}_{3|1,3;2}^{(3)} + \mathbf{C}_{3|2,3;1}^{(3)}\mathbf{u}_{3|2,3;1}^{(3)}\right),
\end{aligned} \tag{3.31}$$

where $\mathbf{H}_{ji}^{(3)} \in \mathbb{C}^{NT^{(3)}\times MT^{(3)}}$ is the channel matrix between Tx_i and Rx_j $i,j \in \{1,2,3\}$.

Choice of $b^{(3)}$ and $T^{(3)}$: $b^{(3)}$ and $T^{(3)}$ are chosen to maximize $\frac{b^{(3)}}{T^{(3)}}$ while ensuring that each of the transmitters can decode the desired order-(2,1) symbols. In the following, we consider the decodability only at Rx_1, where the decodability at other receivers follows due to symmetry. The signal $\mathbf{y}_1^{(3)}$ received by Rx_1 is comprised of 2 known interference terms and 4 terms containing the desired order-(2,1) symbols:

- Known: $\mathbf{H}_{12}^{(3)}\mathbf{C}_{2|2,3;1}^{(3)}\mathbf{u}_{2|2,3;1}^{(3)}$ and $\mathbf{H}_{13}^{(3)}\mathbf{C}_{3|2,3;1}^{(3)}\mathbf{u}_{3|2,3;1}^{(3)}$.
- Desired: $\mathbf{H}_{11}^{(3)}\mathbf{C}_{1|1,2;3}^{(3)}\mathbf{u}_{1|1,2;3}^{(3)}$, $\mathbf{H}_{11}^{(3)}\mathbf{C}_{1|1,3;2}^{(3)}\mathbf{u}_{1|1,3;2}^{(3)}$, $\mathbf{H}_{12}^{(3)}\mathbf{C}_{2|1,2;3}^{(3)}\mathbf{u}_{2|1,2;3}^{(3)}$ and $\mathbf{H}_{13}^{(3)}\mathbf{C}_{3|1,3;2}^{(3)}\mathbf{u}_{3|1,3;2}^{(3)}$.

The desired terms contain in total $4\frac{b^{(3)}}{2} = 2b^{(3)}$ linear combinations of the useful order-(2,1) symbols vectors $\mathbf{u}_{3|2,3;1}^{(3)}$, $\mathbf{u}_{1|1,3;2}^{(3)}$, $\mathbf{u}_{2|1,2;3}^{(3)}$ and $\mathbf{u}_{3|1,3;2}^{(3)}$. To resolve the desired order-(2,1) symbol vectors, the total number $2b^{(3)}$ of the unknowns has to be less than or equal to the size $NT^{(3)}$ of $\mathbf{y}_1^{(3)}$, which is rewritten as a bound

$$B_1^{(3)} \equiv \frac{b^{(3)}}{T^{(3)}} \leq \frac{N}{2}. \tag{3.32}$$

To maximize $\frac{b^{(3)}}{T^{(3)}}$ while fulfilling $B_1^{(3)}$, $b^{(3)}$ and $T^{(3)}$ are chosen as:

$$b^{(3)} = 2N,\; T^{(3)} = 4. \tag{3.33}$$

Similarly to Rx_1, Rx_2 and Rx_3 decode the desired order-(2,1) symbol vectors. With the delivery of the order-(2,1) symbols in phase 3, the transmission is finished and the receivers proceed with the decoding of the desired information symbols.

3.4.2.4 Numbers of Transmission Blocks and Achieved DoF

In this section, we calculate the numbers of transmission blocks and the DoF achieved by the proposed transmission scheme. To ensure all higher-order symbols generated in phase l, $l \in \{1,2,3\}$ can be delivered to the intended receivers, the number $k^{(l)}$ of the transmission blocks of phase l is chosen according to (3.12), where we summarize our calculations for $\frac{3}{4} < \frac{M}{N} \leq \frac{4}{5}$ and $\frac{4}{5} < \frac{M}{N} < 1$ in Tables 3.1 and 3.2, respectively.

Table 3.1: Calculations of k_1, k_2 and k_3 for $\frac{3}{4} < \frac{M}{N} \leq \frac{4}{5}$

Phase 1					Phase 2					Phase 3			
$b^{(1)}$	$T^{(1)}$	$b_\Sigma^{(1)}$	$q_\Sigma^{(1)}$	$k^{(1)}$	$b^{(2)}$	$T^{(2)}$	$b_\Sigma^{(2)}$	$q_\Sigma^{(2)}$	$k^{(2)}$	$b^{(3)}$	$T^{(3)}$	$b_\Sigma^{(3)}$	$k^{(3)}$
$4M$	4	$12M$	$6M$	$3N$	$3N$	4	$6N$	$2N$	$3M$	$2N$	4	$6N$	M

Table 3.2: Calculations of k_1, k_2 and k_3 for $\frac{4}{5} < \frac{M}{N} < 1$

Phase 1					Phase 2					Phase 3			
$b^{(1)}$	$T^{(1)}$	$b_\Sigma^{(1)}$	$q_\Sigma^{(1)}$	$k^{(1)}$	$b^{(2)}$	$T^{(2)}$	$b_\Sigma^{(2)}$	$q_\Sigma^{(2)}$	$k^{(2)}$	$b^{(3)}$	$T^{(3)}$	$b_\Sigma^{(3)}$	$k^{(3)}$
$4N$	5	$12N$	$6N$	3	$3N$	4	$6N$	$2N$	3	$2N$	4	$6N$	1

The achieved DoF are evaluated as

$$d_\Sigma = \frac{3k^{(1)}b^{(1)}}{k^{(1)}T^{(1)} + k^{(2)}T^{(2)} + k^{(3)}T^{(3)}}, \tag{3.34}$$

which yields the DoF given in Theorem 1 for $\frac{3}{4} < \frac{M}{N} \leq \frac{4}{5}$ and $\frac{4}{5} < \frac{M}{N} < 1$, finishing thus the proof for $M < N$.

3.4.3 Case of $M > N$

In this section, we prove Theorem 1 for $M > N$ by describing the DoF achievability scheme employing the RT-PIN concept. The IS-RT-PIN approach is used in phases 1 and 2. In phase 3, the structure of the transmission block is identical that of the scheme proposed by $M < N$.

3.4.3.1 IS-RT Approach

In the this section, we describe the proposed IS-RT approach applied in phases 1 and 2.

Phase 1: A transmission block of phase 1 is split into three parts with the duration of part l, $l \in \{1, 2, 3\}$, of $T^{(1,l)}$ channel uses, where

$$T^{(1,1)} + T^{(1,2)} + T^{(1,3)} = T^{(1)} \tag{3.35}$$

holds. The transmission in parts 1, 2 and 3 is performed as follows.

- Part 1 (IS): all transmitters transmit new information symbols, where from the overheard interference the redundancy to be retransmitted in the next parts is obtained.
- Part 2 (IS and RT): two scheduled transmitters continue to transmit new information symbols, whereas the remaining third transmitter transmits the redundancy generated from the interference terms overheard in part 1.
- Part 3 (RT): all transmitters transmit the redundancy generated from the interference terms overheard in parts 1 and 2.

The scheduling of the transmitters in parts 1, 2 and 3 across the transmission blocks is to ensure that an equal number of information symbols is transmitted by every transmitter in phase 1. For the description of the transmission block, we assume Tx_1 and Tx_2 are scheduled transmit the redundancy only in part 3 and Tx_3 is scheduled to transmit the redundancy in both parts 1 and 2.

Phase 2: A transmission block of phase 2 is split into two parts with the duration of part l, $l \in \{1, 2\}$, of $T^{(2,l)}$ channel uses, where

$$T^{(2,1)} + T^{(2,2)} = T^{(2)} \tag{3.36}$$

holds. For two transmitters scheduled for transmission in the transmission block, the transmission in parts 1 and 2 is performed as follows.

- Part 1 (IS): both transmitters transmit new order-2 symbols.
- Part 2 (RT): one of the transmitters continues to transmit new order-2 symbols, whereas the second transmitter transmits the redundancy generated from the interference terms overheard in part 1.

For the description of the transmission block, we assume a pair of the transmitters $(\mathrm{Tx}_1, \mathrm{Tx}_2)$ is scheduled for the transmission, where Tx_2 is scheduled to transmit the redundancy in part 2.

In the following, we describe the transmission blocks of each phase. In the end, the numbers of the transmission blocks and achieved DoF will be calculated.

3.4.3.2 Phase 1

In a transmission block of phase 1, Tx_1 transmits information symbol vectors $\mathbf{u}_1^{(1,1)} \in \mathbb{C}^{b_1^{(1,1)} \times 1}$ and $\mathbf{u}_1^{(1,2)} \in \mathbb{C}^{b_1^{(1,2)} \times 1}$ and Tx_2 transmits information symbol vectors $\mathbf{u}_2^{(2,1)} \in \mathbb{C}^{b_2^{(2,1)} \times 1}$ and $\mathbf{u}_2^{(2,2)} \in \mathbb{C}^{b_2^{(2,2)} \times 1}$, where $b_1^{(1,1)} = b_2^{(1,1)} = MT^{(1,1)}$ and $b_1^{(1,2)} = b_2^{(1,2)} = MT^{(1,2)}$. Tx_3 transmits the information symbol vector $\mathbf{u}_3^{(1)} \in \mathbb{C}^{b_3^{(1)} \times 1}$, with $b_3^{(1)} \geq NT^{(1,1)}$.

Part 1 (IS): In part 1, Tx_1, Tx_2 and Tx_3 transmit the information symbol vectors $\mathbf{u}_1^{(1,1)}$, $\mathbf{u}_2^{(1,1)}$ and $\mathbf{u}_3^{(1)}$, where the transmitted signals are given by

$$\mathbf{x}_1^{(1,1)} = \mathbf{u}_1^{(1,1)},\ \mathbf{x}_2^{(1,1)} = \mathbf{u}_2^{(1,1)},\ \mathbf{x}_3^{(1,1)} = \mathbf{C}_3^{(1,1)} \mathbf{u}_3^{(1)} \in \mathbb{C}^{MT^{(1,1)} \times 1}, \tag{3.37}$$

with $\mathbf{C}_3^{(1,1)} \in \mathbb{C}^{MT^{(1,1)} \times b_3^{(1)}}$ being the matrix with i.i.d. random precoding coefficients.

By omitting the noise term, the signal received by Rx_j, $j \in \{1, 2\}$, is evaluated as

$$\mathbf{y}_j^{(1,1)} = \mathbf{H}_{j1}^{(1,1)} \mathbf{u}_1^{(1,1)} + \mathbf{H}_{j2}^{(1,1)} \mathbf{u}_2^{(1,1)} + \mathbf{H}_{j3}^{(1,1)} \mathbf{C}_3^{(1,1)} \mathbf{u}_3^{(1)} \in \mathbb{C}^{NT^{(1,1)} \times 1}, \tag{3.38}$$

where $\mathbf{H}_{ji}^{(1,1)} \in \mathbb{C}^{NT^{(1,1)} \times MT^{(1,1)}}$ is the channel matrix between Tx_i and Rx_j.

Part 2 (IS and RT): In part 2, Tx_1 and Tx_2 transmit the information symbol vectors $\mathbf{u}_1^{(1,2)}$ and $\mathbf{u}_2^{(1,2)}$, where the transmitted signals are given by

$$\mathbf{x}_1^{(1,2)} = \mathbf{u}_1^{(1,2)},\ \mathbf{x}_2^{(1,2)} = \mathbf{u}_2^{(1,2)}. \tag{3.39}$$

In part 2, Tx_3 transmits the redundancy generated from the interference terms overheard at Rx_1 and Rx_2 in part 1. The redundancy is chosen such, that the sizes of the receive signal sub-spaces spanned by the transmissions of Tx_3 do not increase at both Rx_1 and Rx_2. For the matrix $\mathbf{V}_{1,2;3}^{(1)} \in \mathbb{C}^{\delta_3 \times b_3^{(1)}}$ used by Tx_3 for the precoding, this can be ensured if the rows of $\mathbf{V}_{1,2;3}^{(1)}$ contain the coefficients of the linear combinations of $\mathbf{u}_3^{(1)}$ which are linear dependent on the linear combinations of $\mathbf{u}_3^{(1)}$ overheard by Rx_1 and Rx_2, written as a condition

$$\text{span}\left(\mathbf{V}_{1,2;3}^{(1)\text{T}}\right) \subseteq \text{span}\left(\left(\mathbf{H}_{13}^{(1,1)} \mathbf{C}_3^{(1,1)}\right)^{\text{T}}\right) \cap \text{span}\left(\left(\mathbf{H}_{23}^{(1,1)} \mathbf{C}_3^{(1,1)}\right)^{\text{T}}\right). \tag{3.40}$$

Since the elements of $\mathbf{H}_{13}^{(1,1)}$, $\mathbf{H}_{23}^{(1,1)}$ and $\mathbf{C}_3^{(1,1)}$ are distributed independently, the size of the intersection sub-space in (3.42) is almost surely $2NT^{(1,1)} - b_3^{(1)}$. In the following, we assume $\mathbf{V}_{1,2;3}^{(1)}$ is full rank with $\delta_3 = 2NT^{(1,1)} - b_3^{(1)}$. The signal transmitted by Tx_3 is given by

$$\mathbf{x}_3^{(2,1)} = \mathbf{C}_3^{(1,2)} \mathbf{V}_{1,2;3}^{(1)} \mathbf{u}_3^{(1)}, \tag{3.41}$$

where $\mathbf{C}_3^{(1,2)} \in \mathbb{C}^{MT^{(1,2)}\times\delta_3}$ is a random matrix with i.i.d. entries.

Part 3 (RT): In part 3, all transmitters transmit the redundancy generated from the interference terms overheard in parts 1 and 2. Similarly to the redundancy transmission given in part 2, the transmitted linear combinations are to be linearly dependent on the linear combinations overheard in parts 1 and 2 by all unintended receivers.

Similarly to the transmission by Tx_3 in part 2, Tx_1 employs full rank precoding matrices $\mathbf{V}_{2,3;1}^{(1,1)} \in \mathbb{C}^{\delta_1^{(1)}\times b_1^{(1,1)}}$ and $\mathbf{V}_{2,3;1}^{(1,2)} \in \mathbb{C}^{\delta_1^{(2)}\times b_1^{(1,2)}}$ for the transmission in part 3 for which

$$\mathrm{span}\left(\mathbf{V}_{2,3;1}^{(1,l)\mathrm{T}}\right) \subseteq \mathrm{span}\left(\mathbf{H}_{21}^{(1,l)\mathrm{T}}\right) \cap \mathrm{span}\left(\mathbf{H}_{31}^{(1,l)\mathrm{T}}\right) \tag{3.42}$$

holds, $\delta_1^{(l)} = 2NT^{(1,l)} - b_1^{(1,l)} = (2N-M)\,T^{(1,l)}$, $l \in \{1,2\}$. The signal transmitted by Tx_1 is given by

$$\mathbf{x}_1^{(1,1)} = \mathbf{C}_1^{(1,3,1)}\mathbf{V}_{2,3;1}^{(1,1)}\mathbf{u}_1^{(1,1)} + \mathbf{C}_1^{(1,3,2)}\mathbf{V}_{2,3;1}^{(1,2)}\mathbf{u}_1^{(1,2)}, \tag{3.43}$$

where $\mathbf{C}_1^{(1,3,1)} \in \mathbb{C}^{MT^{(1,3)}\times\delta_1^{(1)}}$ and $\mathbf{C}_1^{(1,3,2)} \in \mathbb{C}^{MT^{(1,3)}\times\delta_1^{(2)}}$ are random matrices with i.i.d. entries. The precoding matrices used by Tx_2 in part 3 of $\mathbf{V}_{1,3;2}^{(1,1)} \in \mathbb{C}^{\delta_2^{(2)}\times b_2^{(1,1)}}$ and $\mathbf{V}_{1,3;2}^{(1,2)} \in \mathbb{C}^{\delta_2^{(2)}\times b_2^{(1,2)}}$, $\delta_2^{(l)} = \delta_1^{(l)} = (2N-M)\,T^{(1,l)}$, $l \in \{1,2\}$, are defined similarly, where the details are omitted to avoid repetition. The signal transmitted by Tx_2 is given by

$$\mathbf{x}_2^{(1,1)} = \mathbf{C}_2^{(1,3,1)}\mathbf{V}_{1,3;2}^{(1,1)}\mathbf{u}_2^{(1,1)} + \mathbf{C}_2^{(1,3,2)}\mathbf{V}_{1,3;2}^{(1,2)}\mathbf{u}_2^{(1,2)}. \tag{3.44}$$

In part 3, Tx_3 retransmits the terms identical to the ones in part 2, where the transmitted signal is given by

$$\mathbf{x}_3^{(1,3)} = \mathbf{C}_3^{(1,3)}\mathbf{V}_{1,2;3}^{(1)}\mathbf{u}_3^{(1)}, \tag{3.45}$$

with $\mathbf{C}_2^{(1,3,1)} \in \mathbb{C}^{MT^{(1,3)}\times\delta_2^{(1)}}$, $\mathbf{C}_2^{(1,3,2)} \in \mathbb{C}^{MT^{(1,3)}\times\delta_2^{(2)}}$ and $\mathbf{C}_3^{(1,3)} \in \mathbb{C}^{MT^{(1,3)}\times\delta_3}$ being random matrices with i.i.d. entries. By omitting the noise, the signal received by Rx_j, $j \in \{1,2,3\}$, is evaluated as

$$\begin{aligned}\mathbf{y}_j^{(1,3)} = \mathbf{H}_{j1}^{(1,3)}&\left(\mathbf{C}_1^{(1,3,1)}\mathbf{V}_{2,3;1}^{(1,1)}\mathbf{u}_1^{(1,1)} + \mathbf{C}_1^{(1,3,2)}\mathbf{V}_{2,3;1}^{(1,2)}\mathbf{u}_1^{(1,2)}\right) + \\ \mathbf{H}_{j2}^{(1,3)}&\left(\mathbf{C}_2^{(1,3,1)}\mathbf{V}_{1,3;2}^{(1,1)}\mathbf{u}_2^{(1,1)} + \mathbf{C}_2^{(1,3,2)}\mathbf{V}_{1,3;2}^{(1,2)}\mathbf{u}_2^{(1,2)}\right) + \\ &\mathbf{H}_{j3}^{(1,3)}\mathbf{C}_3^{(1,3)}\mathbf{V}_{1,2;3}^{(1)}\mathbf{u}_3^{(1)},\end{aligned} \tag{3.46}$$

where $\mathbf{H}_{ji}^{(1,3)}$ is the channel matrix between Tx_i and Rx_j, $i,j \in \{1,2,3\}$.

PIN: For the overheard interference terms generation, each of the receivers alternatingly cancels the signal of each of the interferers, where the residual interference terms comprise the order-2 symbols to be transmitted in phase 2. We describe the processing at Rx_1. The processing at the other receivers is performed similarly.

First, let us collect the information symbols transmitted by Tx_1 and Tx_2 into the vectors

$$\mathbf{u}_1^{(1)} = \begin{bmatrix}\mathbf{u}_1^{(1,1)\mathrm{T}} & \mathbf{u}_1^{(1,2)\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \mathbf{u}_2^{(1)} = \begin{bmatrix}\mathbf{u}_2^{(1,1)\mathrm{T}} & \mathbf{u}_2^{(1,2)\mathrm{T}}\end{bmatrix}^{\mathrm{T}}. \tag{3.47}$$

The vector of the signals received by Rx_1 of

$$\mathbf{y}_1^{(1)} = \begin{bmatrix}\mathbf{y}_1^{(1,1)\mathrm{T}} & \mathbf{y}_1^{(1,2)\mathrm{T}} & \mathbf{y}_1^{(1,3)\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^{NT^{(1)}\times 1} \tag{3.48}$$

can then be written in a form

$$\mathbf{y}_1^{(1)} = \bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)} + \bar{\mathbf{H}}_{12}^{(1)}\mathbf{u}_2^{(1)} + \bar{\mathbf{H}}_{13}^{(1)}\mathbf{u}_3^{(1)}, \tag{3.49}$$

where $\bar{\mathbf{H}}_{11}^{(1)}, \bar{\mathbf{H}}_{12}^{(1)} \in \mathbb{C}^{NT^{(1)}\times M\left(T^{(1,1)}+T^{(1,2)}\right)}$ and $\bar{\mathbf{H}}_{13}^{(1)} \in \mathbb{C}^{NT^{(1)}\times b_3^{(1)}}$ are the effective channel matrices corresponding to Tx_1, Tx_2 and Tx_3, respectively.

Now, let us consider the nulling of the signal transmitted by Tx_2 in (3.49). Let us express the precoding matrix $\mathbf{V}_{1,3;2}^{(1,l)}$ as product of the channel matrix $\mathbf{H}_{12}^{(1,l)}$ and the projection matrix $\mathbf{V}_{12}^{(1,l)\mathrm{H}} \in \mathbb{C}^{\delta_2^{(1,l)}\times NT^{(1,l)}}$:

$$\mathbf{V}_{1,3;2}^{(1,l)} = \mathbf{V}_{12}^{(1,l)\mathrm{H}}\mathbf{H}_{12}^{(1,l)}, \tag{3.50}$$

$l \in \{1,2\}$. The effective channel matrix between Tx_2 and Rx_1 can then be written as

$$\bar{\mathbf{H}}_{12}^{(1)} = \begin{bmatrix}\mathbf{I}_{NT^{(1,1)}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{NT^{(1,2)}}\\ \mathbf{H}_{12}^{(1,3)}\mathbf{C}_2^{(1,3,1)}\mathbf{V}_{12}^{(1,1)\mathrm{H}} & \mathbf{H}_{12}^{(1,3)}\mathbf{C}_2^{(1,3,2)}\mathbf{V}_{12}^{(1,2)\mathrm{H}}\end{bmatrix}\begin{bmatrix}\mathbf{H}_{12}^{(1,1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{12}^{(1,2)}\end{bmatrix}, \tag{3.51}$$

which is full rank and has $\left(NT^{(1,3)}\right)$-dimensional left null space almost surely. By projecting the received signal as

$$\mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{y}_1^{(1)} = \mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)} + \mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{13}^{(1)}\mathbf{u}_3^{(1)}, \tag{3.52}$$

the signal of Tx_2 is cancelled, where $\mathbf{W}_{12}^{(1)} \in \mathbb{C}^{NT^{(1)}\times NT^{(1,3)}}$ is the matrix, the column of which form a basis in the left null space of $\bar{\mathbf{H}}_{12}^{(1)}$. From the residual interference term in (3.52), a vector of order-2 symbols simultaneously useful for Rx_1 and Rx_3 of $\mathbf{u}_{3|1,3} = \mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{13}^{(1)}\mathbf{u}_3^{(1)} \in \mathbb{C}^{NT^{(1,3)}\times 1}$ is generated.

Following the same steps, the signal of Tx_3 in (3.49) can be cancelled. Let us express $\mathbf{V}_{1,2;3}^{(1)}$ as

$$\mathbf{V}_{1,2;3}^{(1)} = \mathbf{V}_{13}^{(1)\mathrm{H}}\mathbf{H}_{13}^{(1,1)}, \tag{3.53}$$

where $\mathbf{V}_{13}^{(1)\mathrm{H}} \in \mathbb{C}^{\delta_2^{(1,l)}\times NT^{(1,l)}}$ is the projection matrix. The effective channel matrix between Tx_3 and Rx_1 is then evaluated as

$$\bar{\mathbf{H}}_{13}^{(1)} = \begin{bmatrix}\mathbf{I}_{NT^{(1,1)}}\\ \mathbf{H}_{13}^{(1,2)}\mathbf{C}_3^{(1,2)}\mathbf{V}_{13}^{(1)\mathrm{H}}\\ \mathbf{H}_{13}^{(1,3)}\mathbf{C}_3^{(1,3)}\mathbf{V}_{13}^{(1)\mathrm{H}}\end{bmatrix}\mathbf{H}_{13}^{(1,1)}\mathbf{C}_3^{(1,1)}, \tag{3.54}$$

which is full rank and has $\left(N\left(T^{(1,2)}+T^{(1,3)}\right)\right)$-dimensional left null space almost surely. By projecting the received signal as

$$\mathbf{W}_{13}^{(1)\mathrm{H}}\mathbf{y}_1^{(1)} = \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)} + \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{12}^{(1)}\mathbf{u}_2^{(1)}, \tag{3.55}$$

the signal of Tx_3 is cancelled, where $\mathbf{W}_{13}^{(1)} \in \mathbb{C}^{NT^{(1)}\times N\left(T^{(1,2)}+T^{(1,3)}\right)}$ is the matrix, the columns of which form a basis in the left null space of $\bar{\mathbf{H}}_{13}^{(1)}$. From the residual interference term in (3.55), a vector of order-2 symbols simultaneously useful for Rx_1 and Rx_2 of $\mathbf{u}_{2|1,2} = \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{12}^{(1)}\mathbf{u}_2^{(1)} \in \mathbb{C}^{N\left(T^{(1,2)}+T^{(1,3)}\right)\times 1}$ is generated.

Following the same steps, the order-2 symbol vectors $\mathbf{u}_{3|2,3} \in \mathbb{C}^{NT^{(1,3)}\times 1}$ and $\mathbf{u}_{1|1,2} \in \mathbb{C}^{N\left(T^{(1,2)}+T^{(1,3)}\right)\times 1}$ are generated from the residual interference terms at Rx_2 and the order-2 symbol vectors $\mathbf{u}_{2|2,3} \in \mathbb{C}^{NT^{(1,3)}\times 1}$ and $\mathbf{u}_{1|1,3} \in \mathbb{C}^{NT^{(1,3)}\times 1}$ are generated from the residual interference terms at Rx_3, where in total $q^{(1)} = 2N\left(T^{(1,2)}+3T^{(1,3)}\right)$ order-2 symbols are generated in the transmission block.

Choice of $T^{(1,1)}$, $T^{(1,2)}$, $T^{(1)}$ and $b_3^{(1)}$: In order to maximize the achievable DoF, the parameters of the transmission block are chosen to maximize the normalized number $\frac{b_\Sigma^{(1)}}{T^{(1)}}$ of the transmitted information symbols while ensuring the transmitted information symbols are decodable given all generated order-2 symbols are delivered to the receivers which desire them.

First, let us consider the decodability of $\mathbf{u}_1^{(1)}$ at Rx_1. For Rx_1, order-2 symbol vectors $\mathbf{u}_{3|1,3}$, $\mathbf{u}_{2|1,2}$, $\mathbf{u}_{1|1,2}$ and $\mathbf{u}_{1|1,3}$ provide in total $2N\left(T^{(1,2)}+2T^{(1,3)}\right)$ linear combinations. For the decodability, we require the number of the linear combinations to be greater than or equal to the number of the unknowns, which is written as a decodability bound

$$B_3^{(1)} \equiv \frac{4N+M}{4N}\frac{T^{(1,1)}}{T^{(1)}} + \frac{2N+M}{4N}\frac{T^{(1,2)}}{T^{(1)}} \geq 1. \tag{3.56}$$

For Rx_3, order-2 symbol vectors $\mathbf{u}_{3|1,3}$, $\mathbf{u}_{3|2,3}$, $\mathbf{u}_{2|2,3}$ and $\mathbf{u}_{1|1,3}$ provide in total $4NT^{(1,3)}$ linear combinations of $\mathbf{u}_3^{(1)}$, the number of which has to be greater than or equal to the size of $\mathbf{u}_3^{(1)}$: $b_3^{(1)} \leq 4NT^{(1,3)}$. Additionally, $b_3^{(1)} \leq MT^{(1,1)}$ has to hold. A simple calculation shows that $4NT^{(1,3)} \leq MT^{(1,1)}$ for any of the points lying on $B_3^{(1)}$, hence we choose

$$b_3^{(1)} = 4NT^{(1,3)}. \tag{3.57}$$

In the following, we introduce an additional decodability bound which stems from linear independence analysis of the linear combinations used to decode $\mathbf{u}_1^{(1)}$ and $\mathbf{u}_2^{(1)}$ by Rx_1 and Rx_2, respectively.

Lemma 2. $\mathbf{u}_1^{(1)}$ *and* $\mathbf{u}_2^{(1)}$ *are decodable only if the following bound holds:*

$$B_4^{(1)} \equiv \frac{9N-2M}{4N}\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}} \geq 1. \tag{3.58}$$

The proof of Lemma 2 is deferred to Section A.2.

$\frac{b_\Sigma^{(1)}}{T^{(1)}}$ can be expressed in terms of $T^{(1,1)}$, $T^{(1,2)}$ and $T^{(1)}$ as

$$\frac{b_\Sigma^{(1)}}{T^{(1)}} = 4N - 2\left(2N-M\right)\left(\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}}\right), \tag{3.59}$$

which is inversely proportional to $\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}}$. Hence, the maximization of $\frac{b_\Sigma^{(1)}}{T^{(1)}}$ while ensuring decodability is equivalent to the minimization of $\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}}$ subject to the decodability bounds $B_3^{(1)}$ and $B_4^{(1)}$. Depending on whether $B_4^{(1)}$ is active, we distinguish the following two regions of antenna configurations.

1. $1 < \frac{M}{N} \leq \frac{5}{3}$ (Only $B_3^{(1)}$ is active): To minimize $\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}}$ while fulfilling $B_3^{(1)}$, we choose
$$T^{(1,1)} = 4N,\ T^{(1,2)} = 0,\ T^{(1)} = 4N + M, \tag{3.60}$$
where the total number of the generated order-2 symbols is
$$q_\Sigma^{(1)} = 6MN. \tag{3.61}$$

2. $\frac{5}{3} < \frac{M}{N} < 2$ (Both $B_3^{(1)}$ and $B_4^{(1)}$ are active): To minimize $\frac{T^{(1,1)}}{T^{(1)}} + \frac{T^{(1,2)}}{T^{(1)}}$ while fulfilling $B_3^{(1)}$ and $B_4^{(1)}$, we choose
$$T^{(1,1)} = 4N\left(2N-M\right),\ T^{(1,2)} = 4N\left(3N-5M\right), \\ T^{(1)} = 2M^2 - MN - 2N^2, \tag{3.62}$$
where the total number of the generated order-2 symbols is
$$q_\Sigma^{(1)} = 2N\left(6M^2 - 15MN + 10N^2\right). \tag{3.63}$$

3.4.3.3 Phase 2

In phase 2, the order-2 symbols generated in phase 1 are transmitted, where the transmitters are scheduled to transmit in pairs. The transmitters Tx_1 and Tx_2 are scheduled to transmit order-2 symbol vectors $\mathbf{u}_{1|1,2}^{(2,1)} \in \mathbb{C}^{b_1^{(1,1)}\times 1}$, $\mathbf{u}_{1|1,2}^{(2,2)} \in \mathbb{C}^{b_1^{(2,2)}\times 1}$ and $\mathbf{u}_{2|1,2}^{(2)} \in \mathbb{C}^{b_2^{(2)}\times 1}$, where $b_1^{(2,1)} = MT^{(2,1)}$, $b_1^{(2,2)} = MT^{(2,2)}$ and $b_2^{(2)} \geq NT^{(2,1)}$.

Part 1 (IS): In part 1, Tx_1 and Tx_2 transmit order-2 symbols vectors $\mathbf{u}_{1|1,2}^{(2,1)}$ and $\mathbf{u}_{2|1,2}^{(2)}$, respectively, where the transmitted signals are given by

$$\mathbf{x}_1^{(2,1)} = \mathbf{u}_{1|1,2}^{(2,1)},\ \mathbf{x}_2^{(2,1)} = \mathbf{C}_{2|1,2}^{(2,1)}\mathbf{u}_{2|1,2}^{(2,2)}, \tag{3.64}$$

with $\mathbf{C}_{2|1,2}^{(2,1)} \in \mathbb{C}^{MT^{(2,1)}\times b_2^{(2)}}$ being the random matrix with i.i.d. entries. By omitting the noise term, the signal received by Rx_j, $j \in \{1,2,3\}$ is evaluated as

$$\mathbf{y}_j^{(2,1)} = \mathbf{H}_{j1}^{(2,1)}\mathbf{u}_{1|1,2}^{(1)} + \mathbf{H}_{j2}^{(2,1)}\mathbf{C}_{2|1,2}^{(2,1)}\mathbf{u}_{2|1,2}, \tag{3.65}$$

where $\mathbf{H}_{ji}^{(2,1)}$ is the channel matrix between Tx_i and Rx_j, $i \in \{1,2\}$.

Part 2 (RT): In part 2, Tx_1 transmits the order-2 symbol vector $\mathbf{u}_{1|1,2}^{(2,2)}$, where the signal transmitted by Tx_1 writes as

$$\mathbf{x}_1^{(2,1)} = \mathbf{u}_{1|1,2}^{(2,1)}. \tag{3.66}$$

In part 2, Tx_2 transmits the redundancy generated from the interference overheard at Rx_3 in part 1, which ensures the size of the receive signal space spanned by the signals of Tx_2 at Rx_3 does not increase during the transmission in part 2. The transmitted signal is given by

$$\mathbf{x}_2^{(2,2)} = \mathbf{C}_{2|1,2}^{(2,2)}\mathbf{H}_{32}^{(2,1)}\mathbf{C}_{2|1,2}^{(2,1)}\mathbf{u}_{2|1,2}^{(2)}, \tag{3.67}$$

where $\mathbf{C}_{2|1,2}^{(2,2)} \in \mathbb{C}^{MT^{(2,2)}\times NT^{(2,1)}}$ is the random matrix with i.i.d. entries. By omitting the receive noise, the signal received by Rx_j, $j \in \{1,2,3\}$, writes as

$$\mathbf{y}_j^{(2,2)} = \mathbf{H}_{j1}^{(2,2)}\mathbf{u}_{1|1,2}^{(2,2)} + \mathbf{H}_{j2}^{(2,2)}\mathbf{C}_{2|1,2}^{(2,2)}\mathbf{H}_{32}^{(2,1)}\mathbf{C}_{2|1,2}^{(2,1)}\mathbf{u}_{2|1,2}^{(2)}, \tag{3.68}$$

where $\mathbf{H}_{ji}^{(2,2)}$ is the channel matrix between Tx_i and Rx_j, $i \in \{1,2\}$.

PIN: To generate the overheard interference terms, Rx_3 has to cancel the signal of Tx_2 from the received signal.

Let us collect the information symbols transmitted by Tx_1 into a vector

$$\mathbf{u}_1^{(2)} = \begin{bmatrix}\mathbf{u}_1^{(2,1)\mathrm{T}} & \mathbf{u}_1^{(2,2)\mathrm{T}}\end{bmatrix}^{\mathrm{T}}. \tag{3.69}$$

The vector of the signals received by Rx_3 of

$$\mathbf{y}_3^{(2)} = \begin{bmatrix}\mathbf{y}_2^{(2,1)^{\mathrm{T}}} & \mathbf{y}_3^{(2,2)^{\mathrm{T}}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^{NT^{(2)}\times 1} \tag{3.70}$$

can the be written in the form

$$\mathbf{y}_3^{(2)} = \mathbf{H}_{31}^{(2)}\mathbf{u}_{1|1,2}^{(2)} + \bar{\mathbf{H}}_{32}^{(2)}\mathbf{u}_{2|1,2}^{(2)}, \tag{3.71}$$

where $\mathbf{H}_{31}^{(2)}\mathbb{C}^{NT^{(2)}\times MT^{(2)}}$ is the channel matrix between Tx$_1$ and Rx$_3$ and $\bar{\mathbf{H}}_{32}^{(2)} \in \mathbb{C}^{NT^{(2)}\times b_2^{(2)}}$ is the effective channel matrix between Tx$_2$ and Rx$_3$ given by

$$\bar{\mathbf{H}}_{32}^{(2)} = \begin{bmatrix} \mathbf{I}_{NT^{(2,1)}} \\ \mathbf{H}_{32}^{(2,2)}\mathbf{C}_{2|1,2}^{(2,2)} \end{bmatrix} \mathbf{H}_{32}^{(2,1)}\mathbf{C}_{2|1,2}^{(2,1)}, \tag{3.72}$$

which is full rank and has $\left(NT^{(2,2)}\right)$-dimensional left null space almost surely. By projecting the received signal as

$$\mathbf{W}_3^{(2)\mathrm{H}}\mathbf{y}_3^{(2)} = \mathbf{W}_3^{(2)\mathrm{H}}\mathbf{H}_{31}^{(2)}\mathbf{u}_{1|1,2}^{(2)}, \tag{3.73}$$

the interference signal of Tx$_2$ is cancelled, where $\mathbf{W}_3^{(2)} \in \mathbb{C}^{NT^{(2)}\times NT^{(2,2)}}$ is the matrix whose columns form a basis in the left null space of $\bar{\mathbf{H}}_{32}^{(2)}$. From (3.73), the order-(2,1) symbol vector $\mathbf{u}_{1|1,2;3} = \mathbf{W}_3^{(2)\mathrm{H}}\mathbf{H}_{31}^{(2)}\mathbf{u}_{1|1,2}^{(2)} \in \mathbb{C}^{NT^{(2,2)}\times 1}$ is generated with the total number of generated order-(2,1) symbols $q_\Sigma^{(2)} = NT^{(2,2)}$.

Choice of $T^{(2,1)}$, $T^{(2)}$ and $b_2^{(2)}$: The parameters of the transmission block have to be chosen such that the normalized number of the transmitted order-2 symbols $\frac{b_\Sigma^{(2)}}{T^{(2)}}$ is maximized while ensuring the decodability of $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$ given all generated order-(2,1) symbols are delivered to Rx$_1$ and Rx$_2$. Since $b_1^{(2)} = MT^{(2)}$, maximizing $\frac{b_\Sigma^{(2)}}{T^{(2)}}$ is equivalent to maximizing $\frac{b_2^{(2)}}{T^{(2)}}$.

For the decodability, we first introduce the following two decodability bounds.

- The number $N\left(T^{(2)} + T^{(2,2)}\right)$ of the linear combinations of $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$ provided to Rx$_i$ by $\mathbf{y}_i^{(2)}$ and $\mathbf{u}_{1|1,2;3}$, $i \in \{1,2\}$, has to be greater than or equal to the number of the unknowns $MT^{(2)} + b_2^{(2)}$, rewritten as a bound

$$B_3^{(2)} \equiv \frac{b_2^{(2)}}{T^{(2)}} \leq (2N - M) - N\frac{T^{(2,1)}}{T^{(2)}}. \tag{3.74}$$

- The number $b_2^{(2)}$ of the order-2 symbols transmitted by Tx$_2$ has to be less than or equal to the total number of the transmit signal dimensions $MT^{(2,1)}$, rewritten as a bound

$$B_4^{(2)} \equiv \frac{b_2^{(2)}}{T^{(2)}} \leq M\frac{T^{(2,1)}}{T^{(2)}}. \tag{3.75}$$

The following lemma givens an additional bound which stems from linear independence analysis of the linear combinations used to decode $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$ by Rx$_1$ and Rx$_2$.

Lemma 3. $\mathbf{u}_{1|1,2}^{(2)}$ *and* $\mathbf{u}_{2|1,2}^{(2)}$ *are decodable only if the following bound holds:*

$$B_5^{(2)} \equiv \frac{T^{(2,1)}}{T^{(2)}} \geq \frac{2N-M}{4N-M}. \tag{3.76}$$

The proof of Lemma 3 is deferred to section A.3.

To maximize $\frac{b_2^{(2)}}{T^{(2)}}$ while satisfying $B_3^{(2)}$, $B_4^{(2)}$ and $B_5^{(2)}$, we first ensure $B_3^{(2)}$ is always fulfilled by setting

$$\frac{b_2^{(2)}}{T^{(2)}} = (2N-M) - N\frac{T^{(2,1)}}{T^{(2)}}, \tag{3.77}$$

where $B_4^{(2)}$ can then be rewritten in terms of $\frac{T^{(2,1)}}{T^{(2)}}$ as

$$B_4^{(2)*} \equiv \frac{T^{(2,1)}}{T^{(2)}} \geq \frac{2N-M}{M+N}. \tag{3.78}$$

Since $\frac{b_2^{(2)}}{T^{(2)}}$ is inversely proportional to $\frac{T^{(2,1)}}{T^{(2)}}$, to maximize $\frac{b_2^{(2)}}{T^{(2)}}$, $\frac{T^{(2,1)}}{T^{(2)}}$ is chosen as minimum satisfying both $B_4^{(2)*}$ and $B_5^{(2)}$. Depending on whether $B_4^{(2)*}$ or $B_5^{(2)}$ are active, the following two regions of antenna configurations are distinguished.

1. $1 < \frac{M}{N} \leq \frac{3}{2}$ ($B_4^{(2)*}$ is active): To fulfil $B_4^{(2)*}$, we choose

   $$T^{(2,1)} = 2N-M,\ T^{(2)} = M+N, \tag{3.79}$$

   where the total number of the generated order-(2,1) symbols is given by

   $$q_{\Sigma}^{(2)} = N\left(2N-M\right). \tag{3.80}$$

2. $\frac{3}{2} < \frac{M}{N} < 2$ ($B_5^{(2)}$ is active): To fulfil $B_5^{(2)}$, we choose

   $$T^{(2,1)} = 2N-M,\ T^{(2)} = 4N-M, \tag{3.81}$$

   where the total number of the generated order-(2,1) symbols is given by

   $$q_{\Sigma}^{(2)} = 2N^2. \tag{3.82}$$

3.4.3.4 Phase 3

The structure of a transmission block of phase 3 is identical to the one given in Section 3.4.2.3 with $T^{(3)} = 4N$ and $b_1^{(3)} = b_2^{(3)} = b_3^{(3)} = 2N$.

3.4.3.5 Numbers of Transmission Blocks and Achieved DoF

In this section the numbers of the transmission blocks for each phase and the DoF achieved by the transmission scheme are evaluated.

The calculations of the numbers of the transmission blocks k_l, $l \in \{1, 2, 3\}$, fulfilling (3.12) are given in Tables 3.3, 3.4 and 3.5. The achieved DoF are evaluated as

$$d_\Sigma = \frac{k^{(1)} b_\Sigma^{(1)}}{k^{(1)} T^{(1)} + k^{(2)} T^{(2)} + k^{(3)} T^{(3)}}, \tag{3.83}$$

which yields the DoF given in Theorem 1 for $1 < \frac{M}{N} \leq \frac{3}{2}$, $\frac{3}{2} < \frac{M}{N} \leq \frac{5}{3}$ and $\frac{5}{3} < \frac{M}{N} < 2$, finishing thus the proof for $M > N$.

3.5 Conclusion

In this chapter, the three-user symmetric MIMO IC with delayed CSIT has been considered. For the considered scenario, two novel transmission schemes achieving a number of DoF greater than that in the literature were proposed. Additionally, an upper bound on the number of DoF assuming linear coding strategies was provided, shown to be tight for a particular region of antenna configurations.

Table 3.3: Calculations of k_1, k_2 and k_3 for $1 < \frac{M}{N} \leq \frac{3}{2}$

Phase 1				Phase 2				Phase 3		
$b_\Sigma^{(1)}$	$T^{(1)}$	$q^{(1)}$	k_1	$b_\Sigma^{(2)}$	$T^{(2)}$	$q^{(2)}$	k_2	$b_\Sigma^{(3)}$	$T^{(3)}$	k_3
$12MN$	$6MN$	$4N+M$	3	$3MN$	$M+N$	$N(2M-N)$	6	$6N$	$4N$	$2M-N$

Table 3.4: Calculations of k_1, k_2 and k_3 for $\frac{3}{2} < \frac{M}{N} \leq \frac{5}{3}$

Phase 1				Phase 2				Phase 3		
$b_\Sigma^{(1)}$	$T^{(1)}$	$q^{(1)}$	k_1	$b_\Sigma^{(2)}$	$T^{(2)}$	$q^{(2)}$	k_2	$b_\Sigma^{(3)}$	$T^{(3)}$	k_3
$12MN$	$6MN$	$4N+M$	$6N-M$	$N(6N-M)$	$4N-M$	$2N^2$	$6M$	$6N$	$4N$	$2MN$

Table 3.5: Calculations of k_1, k_2 and k_3 for $\frac{5}{3} < \frac{M}{N} < 2$

Phase 1				Phase 2				Phase 3		
$b_\Sigma^{(1)}$	$T^{(1)}$	$q^{(1)}$	k_1	$b_\Sigma^{(2)}$	$T^{(2)}$	$q^{(2)}$	k_2	$b_\Sigma^{(3)}$	$T^{(3)}$	k_3
$4N(6M^2-15MN+10N^2)$	$2M^2-MN-2N^2$	$2N(6M^2-15MN+10N^2)$	$3(6N-M)$	$N(6N-M)$	$4N-M$	$2N^2$	$6(6M^2-15MN+10N^2)$	$6N$	$4N$	$2N(6M^2-15MN+10N^2)$

Chapter 4

The DoF of the 2-Antenna 3-User MISO Broadcast Channel with Alternating CSIT

4.1 Introduction

In this chapter, the 2-antenna 3-user MISO BC is considered. We assume, that the CSIT for each user can be either perfect (P) or delayed (D), resulting thus in total in 8 possible CSIT states $I_1I_2I_3$, $I_i \in \{P, D\}$, $i \in \{1, 2, 3\}$. As we already mentioned in Section 1.1, performing joint encoding over different CSIT states brings DoF gains as compared to encoding over each CSIT state independently. While the DoF of the 2-user MISO BC with alternating CSIT are well studied, the DoF characterization for the K-user case is yet an open question.

Having $M < K$ makes the problem of DoF characterization yet more challenging. Nevertheless, a number of works achieved optimal DoF for the case $M = 2$, $K = 3$ for particular CSIT configurations. For purely delayed CSIT, [MAT12] achieved optimal DoF by using MAT. For alternating CSIT, existing works obtained DoF characterizations by achieving the outer bounds provided in [TJSSP13] and [CYE13]. By using ZF, [TJSSP13] achieved optimal DoF for the case where at least for M users perfect CSIT is available. [CYE13] characterized optimal DoF for the setting in which the CSIT alternates between the states assumed in [TJSSP13] and the jointly delayed DDD state. [LH14] achieved optimal DoF by performing joint encoding over the jointly perfect PPP state and the jointly delayed DDD state. [ATS14] achieved optimal DoF for the fixed hybrid PDD state.

In this chapter, we obtain two new results on the DoF characterization for the $M = 2$, $K = 3$ MISO BC with alternating CSIT. The first result establishes the DoF region for the CSIT setting, where the admissible CSIT states can take the following 5 values: PPP, PPD, PDP, PDD and DDD. It the following, we refer to such setting as restricted alternating CSIT. The second result characterizes the DoF for the case where the CSIT states can take all possible values, however the joint CSIT state probabilities are restricted to fulfil certain relationships. Our results generalize the results existing in the literature by achieving optimal DoF for a broader range of CSIT configurations. For the outer bound, we rely on the existing outer bound for delayed and imperfect CSIT in [CYE13]. For the achievability, we introduce four novel CSs which perform joint encoding over the CSIT state tuples (PPP, PDD), (PDD, DDD), (PDD, DPD, DDD) and (PDD, DPD, DDP). By assigning the newly proposed and existing CSs to the available CSIT states, optimal DoF are achieved. Parts of the results of this chapter has been published by the author of this thesis in [BK19].

The rest of the chapter is organized as follows. The system model will be introduced in Section 4.2. The main results will be given in Section 4.3. The CSs necessary for the DoF characterization will be introduced in Section 4.4. In Section 4.5, the proof of the DoF characterization result for the restricted alternating CSIT will be given. In Section 4.6, we give the proof of the DoF characterization result for the case where the CSIT states can take all possible values. The conclusions of the chapter will be given in Section 4.7.

4.2 System Model

In this chapter, the 2-antenna 3-user MISO BC depicted in Fig. 4.1 is considered, which is comprised of the two-antenna transmitter Tx and three single-antenna receivers Rx_i, $i \in \{1, 2, 3\}$. The signal received by Rx_i at the t-th channel use is given by

$$y_i(t) = \mathbf{h}_i^{\mathrm{T}}(t)\,\mathbf{x}(t) + z_i(t), \tag{4.1}$$

where $\mathbf{x}(t) \in \mathbb{C}^{2\times 1}$ is the vector of the transmitted signals, $\mathbf{h}_i(t) \in \mathbb{C}^{2\times 1}$ is the vector of the channel coefficients between Tx and Rx_i, and $z_i(t) \sim \mathcal{CN}(0,1)$ is the additive white Gaussian noise at Rx_i. The transmitted signal is subject to the average transmit power constraint $\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}\left\{\|\mathbf{x}(t)\|^2\right\} \le P$, where n is the communication duration. Channel coefficients are drawn from continuous distributions and are independent across transmit antennas, receivers and different channel uses.

In this chapter, an alternating CSIT setting is considered, which is defined as follows. At each time instant, the knowledge about the channel coefficients corresponding to Rx_i, $i \in \{1, 2, 3\}$, can be either perfect (P) or delayed (D). In total, there are $2^3 = 8$ possible CSIT states denoted by $I_1 I_2 I_3$, $I_i \in \{\mathrm{P}, \mathrm{D}\}$, $i \in \{1, 2, 3\}$. The probability of the state $I_1 I_2 I_3$ is called the joint CSIT state probability and is denoted by $\lambda_{I_1 I_2 I_3}$, where $\sum_{I_1 I_2 I_3} \lambda_{I_1 I_2 I_3} = 1$. The probability that the CSIT for Rx_i is perfect is called marginal CSIT state probability and is given by $\lambda_i = \sum_{I_1 I_2 I_3, I_i = P} \lambda_{I_1 I_2 I_3}$. Without loss of generality, we assume the users to be ordered such that $\lambda_1 \ge \lambda_2 \ge \lambda_3$ holds. At every receiver, global instantaneous CSI is assumed.

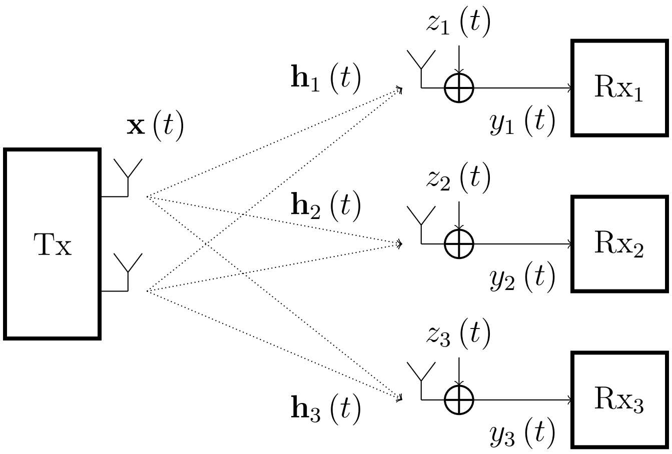

Figure 4.1: 2-antenna 3-user MISO BC

We assume that Tx has messages to each receiver with W_i, $i \in \{1,2,3\}$, denoting the message intended to Rx_i. The message W_i is assumed to be uniformly distributed on the message set $\mathcal{W}_i$. The rate for Rx_i in bits per channel use is given by $R_i(P) = \frac{\log_2 |\mathcal{W}_i|}{n}$, where the rate tuple is denoted by $\mathbf{R}(P) = (R_1(P), R_2(P), R_3(P)) \in \mathbb{R}^3$. A $\left(2^{nR_1(P)}, 2^{nR_2(P)}, 2^{nR_3(P)}, n\right)$ code of length n and rate $\mathbf{R}(P)$ consists of:

1. the message sets $\mathcal{W}_i$, $i \in \{1,2,3\}$,
2. the sets of encoding functions $\{\phi_i(t)\}_{t=1}^{n}$, which map every message $W_i \in \mathcal{W}_i$, $i \in \{1,2,3\}$, to the transmitted signals $\mathbf{x}(t)$, $1 \le t \le n$,
3. the decoding functions $\psi_j(t)$, which map the set of the received signals $\{y_j(t)\}_{t=1}^{n}$ to the decoded message $\hat{W}_j \in \mathcal{W}_j$, $j \in \{1,2,3\}$.

For a given power constraint P, the rate tuple $\mathbf{R}(P)$ is said to be achievable if there exists a sequence of $\left(2^{nR_1(P)}, 2^{nR_2(P)}, 2^{nR_3(P)}, n\right)$ codes for which the decoding error probability $P_e = \mathbb{P}\left((W_1, W_2, W_3) \neq \left(\hat{W}_1, \hat{W}_2, \hat{W}_3\right)\right)$ goes to zero as soon as $n \to \infty$. For a given power constraint P, the capacity region $\mathcal{C}(P)$ is defined as the closure of the set of all achievable rate tuples $\mathbf{R}(P)$ for which the power constraint holds.

We say that the DoF tuple (d_1, d_2, d_3) is achievable, if there exists a sequence of achievable rate tuples $\mathbf{R}(P) \in \mathcal{C}(P)$, such that $d_i = \lim_{P\to\infty} \frac{R_i(P)}{\log_2(P)}$, $i \in \{1,2,3\}$. The closure of the set of all achievable DoF tuples is called the DoF region $\mathcal{D}$. The achievable sum-DoF is denoted by $d_\Sigma = d_1 + d_2 + d_3$, where the maximum achievable sum-DoF (or simply DoF) is given by

$$d = \max_{(d_1,d_2,d_3)\in\mathcal{D}} d_1 + d_2 + d_3. \tag{4.2}$$

4.3 Main Results

In the this section, we state the main results of the chapter.

First, we introduce the following alternating CSIT setting.

Definition 1. *The alternating CSIT setting where*

$$\lambda_{\text{DPP}} = \lambda_{\text{DPD}} = \lambda_{\text{DDP}} = 0, \tag{4.3}$$

i.e. where the admissible CSIT states comprise the set $\{\text{PPP}, \text{PPD}, \text{PDP}, \text{PDD}, \text{DDD}\}$*, is called the restricted alternating CSIT.*

For the restricted alternating CSIT setting, we state the first result of the chapter.

Theorem 3. *The DoF region of the 2-antenna 3-user MISO BC with restricted alternating CSIT is given by*

$$\begin{aligned} A_1 &\equiv 2d_1 + d_2 + d_3 \leq 2 + \lambda_1, \\ A_2 &\equiv d_1 + 2d_2 + d_3 \leq 2 + \lambda_2, \\ A_3 &\equiv d_1 + d_2 + 2d_3 \leq 2 + \lambda_3, \\ B &\equiv d_1 + d_2 + d_3 \leq 2, \\ C_1 &\equiv d_1 \leq 1, \\ C_2 &\equiv d_2 \leq 1, \\ C_3 &\equiv d_3 \leq 1. \end{aligned} \tag{4.4}$$

Proof. For the outer bound, we refer to the outer bound in [CYE13] for delayed and imperfect CSIT, in which the CSIT quality exponents $\alpha_i(t) = 0$ and $\alpha_i(t) = 1$ are equivalent in the DoF sense in the alternating CSIT setting to the states $I_i = \text{D}$ and $I_i = \text{P}$, respectively. As for the achievability, the set of necessary CSs will be introduced in Section 4.4 and the formal proof will be provided in Section 4.5.

□

From the result given by Theorem 3, the following corollary follows.

Corollary 2. *The DoF of the 2-antenna 3-user MISO BC with restricted alternating CSIT are*

$$d = \begin{cases} \frac{3}{2} + \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3) & \text{if} \begin{array}{c} 3\lambda_1 - \lambda_2 - \lambda_3 \leq 2, \\ \lambda_1 + \lambda_2 + \lambda_3 \leq 2, \end{array} \\ \frac{5}{3} + \frac{1}{3}(\lambda_2 + \lambda_3) & \text{if} \quad 3\lambda_1 - \lambda_2 - \lambda_3 > 2, \\ 2 & \text{if} \quad \lambda_1 + \lambda_2 + \lambda_3 > 2. \end{cases} \tag{4.5}$$

Below, we provide the analysis of the optimal DoF tuples for each of the cases given in (4.5).

1. Region I: $3\lambda_1 - \lambda_2 - \lambda_3 \leq 2, \lambda_1 + \lambda_2 + \lambda_3 \leq 2$.

In this case, the bound B is inactive. The optimal DoF tuple $\mathbf{A}_{\text{I}} = \left(\frac{1}{2} + \frac{3\lambda_1 - \lambda_2 - \lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_2 - \lambda_1 - \lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_3 - \lambda_1 - \lambda_2}{4}\right)$ is an intersection of bounds A_1, A_2 and A_3.

2. Region II: $3\lambda_1 - \lambda_2 - \lambda_3 > 2$.

In this case, the bounds A_1 and B are inactive. The optimal DoF tuple $\mathbf{A}_{\text{II}} = \left(1, \frac{1}{3} + \frac{2\lambda_2 - \lambda_3}{3}, \frac{1}{3} + \frac{2\lambda_3 - \lambda_2}{3}\right)$ is an intersection of bounds C_1, A_2 and A_3.

3. Region III: $\lambda_1 + \lambda_2 + \lambda_3 > 2$.

In this case, all bounds are active. The are three optimal DoF tuples which are the corner ponts of the DoF region $\mathbf{A}_{\text{III}}^{[1]} = (\lambda_1, \lambda_2, 2 - \lambda_1 - \lambda_2)$, $\mathbf{A}_{\text{III}}^{[2]} = (\lambda_1, 2 - \lambda_1 - \lambda_3, \lambda_3)$ and $\mathbf{A}_{\text{III}}^{[3]} = (2 - \lambda_2 - \lambda_3, \lambda_2, \lambda_3)$, which correspond to the intersections of the bound B with the pairs of bounds (A_1, A_2), (A_1, A_3) and (A_2, A_3), respectively.

The forms of the DoF regions will be studied in detail in Section 4.5.2.

The second result considers the setting where the admissible states can take all possible values, for which the following result is obtained.

Theorem 4. *For the 2-antenna 3-user MISO BC with alternating CSIT, the DoF are given by*

$$d = \begin{cases} \frac{3}{2} + \frac{1}{4}\left(\lambda_1 + \lambda_2 + \lambda_3\right) & \text{if} \quad \begin{array}{c} \lambda_1 + \lambda_2 + \lambda_3 \leq 2, \\ \lambda_{\mathrm{PDD}} + \lambda_{\mathrm{DPD}} + \lambda_{\mathrm{DDP}} - \lambda_{\mathrm{PPP}} \leq 2\lambda_{\mathrm{DDD}} \end{array} \\ 2 & \text{if} \quad \lambda_1 + \lambda_2 + \lambda_3 > 2, \\ \frac{12}{7} & \text{if} \quad \left(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDD}}\right) = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{7}\right) \\ \frac{33}{19} & \text{if} \quad \left(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDP}}\right) = \left(\frac{15}{19}, \frac{2}{19}, \frac{2}{19}\right). \end{cases} \tag{4.6}$$

The proof of Theorem 4 will be given in Section 4.6.

Remark 1. *Theorem 4 establishes the optimal DoF fully for Region III. Optimal DoF are established also fully for Region I when the alternating CSIT setting is restricted since* $\lambda_{\mathrm{PDD}} + \lambda_{\mathrm{DPD}} + \lambda_{\mathrm{DDP}} - \lambda_{\mathrm{PPP}} \leq 2\lambda_{\mathrm{DDD}}$ *is equivalent to* $3\lambda_1 - \lambda_2 - \lambda_3 \leq 2$ *when* (4.3) *holds.*

4.4 Constituent Encoding Schemes

In this section, the set of CSs necessary to prove Theorems 3 and 4 is introduced. In each of the following sections, schemes achieving particular numbers of DoF will be described.

4.4.1 Schemes Achieving 2 DoF

In this section, CSs achieving 2 DoF are described.

S_1^2, S_2^2, S_3^2 and S_4^2 perform individual encoding over PPD, PDP, DPP and PPP states, respectively. S_1^2, S_2^2, S_3^2 and S_4^2 are the well known ZF-based schemes in which two symbols are transmitted to two receivers over a single channel use. S_5^2 is a first newly proposed transmission scheme, in which joint encoding over PPP and PDD states is performed. By swapping the receiver indices in S_5^2, the schemes S_6^2 and S_7^2 which performing joint encoding over the CSIT state pairs (PPP, DPD) and (PPP, DDP), respectively, are obtained. S_8^2 is the scheme known from [LH14], in which joint encoding over PPP and DDD states is performed. The details of the 2 DoF achieving CSs which exist in the literature are given in Table 4.1. The details of the newly proposed CS S_5^2 are provided in the following.

Table 4.1: Summary of the constituent encoding schemes achieving 2 DoF.

CS	State fractions	DoF tuples	Achievability
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1, 1, 0)$	ZF
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1, 0, 1)$	ZF
S_3^2	$\lambda_{\mathrm{DPP}} = 1$	$(0, 1, 1)$	ZF
S_4^2	$\lambda_{\mathrm{PPP}} = 1$	$(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$	ZF
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	Novel scheme
S_6^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DPD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1, \frac{1}{2}\right)$	-
S_7^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDP}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 1\right)$	-
S_8^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	[LH14]

S_5^2 achieves the DoF tuple $(d_1, d_2, d_3) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ for the CSIT state fractions $(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ by encoding over two channel uses: $t = 1$ corresponding to the state PDD and $t = 2$ corresponding to the state PPP. During the transmission, two symbols $u_1^{[1]}$ and $u_1^{[2]}$ are delivered to Rx_1, the symbol u_2 is delivered to Rx_2 and the symbol u_3 is delivered to Rx_3.

At $t = 1$, the symbol vector $\mathbf{u}_1 = [u_1^{[1]} \quad u_1^{[2]}]^{\mathrm{T}}$ is transmitted using random precoding and the symbols u_2 and u_3 are transmitted using ZF to ensure that no interference is overheard by Rx_1. The signal transmitted at $t = 1$ is given by

$$\mathbf{x}(1) = \mathbf{C}_1(1)\,\mathbf{u}_1 + \mathbf{c}_{23}(1)\,(u_2 + u_3)\,, \tag{4.7}$$

where $\mathbf{C}_1(1) \in \mathbb{C}^{2\times 2}$ is a random matrix with independent entries taken from continuous distributions and $\mathbf{c}_{23} \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1) = 0$. By omitting the receive noise, the signal received by Rx_1 is

$$y_1(1) = \mathbf{h}_1^{\mathrm{T}}(1)\,\mathbf{C}_1(1)\,\mathbf{u}_1 \tag{4.8}$$

and for Rx_2 and Rx_3 the received signals are

$$y_i(1) = \mathbf{h}_i^{\mathrm{T}}(1)\,(\mathbf{C}_1(1)\,\mathbf{u}_1 + \mathbf{c}_{23}(1)\,(u_2 + u_3))\,, \tag{4.9}$$

$i \in \{2, 3\}$.

At $t = 2$, ZF is applied for the transmission to Rx_2 and Rx_3 to ensure that no interference is overheard by Rx_1. Additionally, perfect CSIT available for Rx_2 and Rx_3 is employed to design the precoding vectors such that at $t = 2$, Rx_2 and Rx_3 overhear the interference identical to that at $t = 1$.

The signal transmitted at $t = 2$ is given by

$$\mathbf{x}(2) = \mathbf{C}_1(2)\,\mathbf{u}_1 + \mathbf{c}_{23}(2)\left(\gamma_2 u_2 + \gamma_3 u_3\right), \tag{4.10}$$

where $\mathbf{c}_{23}(2) \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(2)\,\mathbf{c}_{23}(2) = 0$. $\mathbf{C}_1(2) \in \mathbb{C}^{2\times 2}$ is the precdoing matrix which is to fulfil

$$\begin{bmatrix}\mathbf{h}_2^{\mathrm{T}}(2)\\ \mathbf{h}_3^{\mathrm{T}}(2)\end{bmatrix}\mathbf{C}_1(2) = \begin{bmatrix}\mathbf{h}_2^{\mathrm{T}}(1)\\ \mathbf{h}_3^{\mathrm{T}}(1)\end{bmatrix}\mathbf{C}_1(1), \tag{4.11}$$

which is ensured by setting

$$\mathbf{C}_1(2) = \begin{bmatrix}\mathbf{h}_2^{\mathrm{T}}(2)\\ \mathbf{h}_3^{\mathrm{T}}(2)\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{h}_2^{\mathrm{T}}(1)\\ \mathbf{h}_3^{\mathrm{T}}(1)\end{bmatrix}\mathbf{C}_1(1). \tag{4.12}$$

$\gamma_2, \gamma_3 \in \mathbb{C}$ are the precoding scalars which have to fulfil

$$\mathbf{h}_3^{\mathrm{T}}(2)\,\mathbf{c}_{23}(2)\,\gamma_2 = \mathbf{h}_3^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1), \tag{4.13}$$

$$\mathbf{h}_2^{\mathrm{T}}(2)\,\mathbf{c}_{23}(2)\,\gamma_3 = \mathbf{h}_2^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1), \tag{4.14}$$

which is ensured by setting

$$\gamma_2 = \frac{\mathbf{h}_3^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1)}{\mathbf{h}_3^{\mathrm{T}}(2)\,\mathbf{c}_{23}(2)}, \gamma_3 = \frac{\mathbf{h}_2^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1)}{\mathbf{h}_2^{\mathrm{T}}(2)\,\mathbf{c}_{23}(2)}. \tag{4.15}$$

At $t = 1, 2$, Rx_1 receives two linear combinations of the elements of $\mathbf{u}_1$ which allows to decode the symbol vector. Rx_2 and Rx_3 cancel the interference in the received signals as $y_j(1) - y_j(2)$, $j \in \{2, 3\}$, where from the obtained interference-free signal, u_2 and u_3 are decoded.

4.4.2 Schemes Achieving $\frac{5}{3}$ DoF

In this section, transmission schemes achieving $\frac{5}{3}$ DoF are described.

$S_1^{5/3}$ is the second newly proposed transmission scheme, in which joint encoding over PDD and DDD states is performed. By swapping the receiver indices in $S_1^{5/3}$, the schemes $S_2^{5/3}$ and $S_3^{5/3}$ are obtained, in which joint encoding over the CSIT state pairs (DPD, DDD) and (DDP, DDD), respectively, is performed. $S_4^{5/3}$ is a transmission

Table 4.2: Summary of the constituent encoding schemes achieving $\frac{5}{3}$ DoF.

CS	State fractions	DoF tuples	Achievability
$S_1^{5/3}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	Novel scheme
$S_2^{5/3}$	$(\lambda_{\text{DPD}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, 1, \frac{1}{3}\right)$	-
$S_3^{5/3}$	$(\lambda_{\text{DDP}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, \frac{1}{3}, 1\right)$	-
$S_4^{5/3}$	$(\lambda_{\text{PPD}}, \lambda_{\text{PDP}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(0, 1, \frac{2}{3}\right)$	[TJSSP13]

scheme for achieving a non-optimal DoF region corner point, in which joint encoding over PDP and PPD states is performed. For $S_4^{5/3}$, we refer to the transmission scheme for the 2-user MISO BC given in [TJSSP13], which jointly encodes over PD and DP states and achieves the DoF tuple $(d_1, d_2) = \left(1, \frac{2}{3}\right)$. The details of the $\frac{5}{3}$ DoF achieving CSs which exist in the literature are given in Table 4.2. The details of the newly proposed CS $S_1^{5/3}$ are provided in the following.

The achievability of $\frac{5}{3}$ DoF for the fixed PDD setting is known from [ATS14]. By substituting the PDD states which do not require perfect CSIT with DDD states, the achievability in [ATS14] can be extended to the setting with $(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{5}{6}, \frac{1}{6}\right)$. The scheme $S_1^{5/3}$ improves upon the scheme in [ATS14] by achieving $\frac{5}{3}$ DoF for the setting with $(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$. The DoF tuple $(d_1, d_2, d_3) = \left(1, \frac{1}{3}, \frac{1}{3}\right)$ is achieved.

In $S^{5/3}$, joint encoding over 6 PDD and 3 DDD states is performed. During the transmission, nine symbols $\{u_1^{[k]}\}_{k=1}^{9}$ are delivered to Rx_1, three symbols $\{u_2^{[k]}\}_{k=1}^{3}$ are delivered to Rx_2 and three symbols $\{u_3^{[k]}\}_{k=1}^{3}$ are delivered to Rx_3.

The transmission is split into two phases. Phase 1 comprises the first five channel uses $t = 1, 2, 3, 4, 5$, each having a PDD state, during which the original information symbols are transmitted. From the interference terms overheard in phase 1, five order-2 symbols $u_{2,3}$, $u_{1,2}^{[1]}$, $u_{1,2}^{[2]}$, $u_{1,3}^{[1]}$ and $u_{1,3}^{[2]}$ are generated. The transmission of the generated order-2 symbols is performed in phase 2 which comprises the remaining four channel uses $t = 6, 7, 8, 9$, three of which have DDD state and one has PDD state. The summary of the transmission in phases 1 and 2 is given in Table 4.3, where the overheard interference terms are marked by red. The detailed description of the transmission is provided below.

Table 4.3: Summary of the scheme $S_1^{5/3}$

t	State	Rx$_1$	Rx$_2$	Rx$_3$	Generated Symbol
1	PDD	$\beta_1^{[1]}u_1^{[1]}$	$\beta_1^{[1]}u_2^{[1]}+$ $L_2^{[1]}(u_1^{[1]}, u_3^{[1]})$	$\beta_2^{[1]}u_3^{[1]}+$ $L_3^{[1]}(u_1^{[1]}, u_2^{[1]})$	$u_{2,3}$ as in (4.21)
2	PDD	$L_1^{[1]}(\mathbf{u}_1^{[2,3]})$	$\beta_2^{[4]}u_2^{[2]}+$ $L_2^{[2]}(\mathbf{u}_1^{[2,3]})$	-	$u_{1,2}^{[1]} =$ $L_2^{[2]}(\mathbf{u}_1^{[2,3]})$
3	PDD	$L_1^{[2]}(\mathbf{u}_1^{[4,5]})$	$\beta_2^{[5]}u_2^{[3]}+$ $L_2^{[3]}(\mathbf{u}_1^{[4,5]})$	-	$u_{1,2}^{[2]} =$ $L_2^{[3]}(\mathbf{u}_1^{[4,5]})$
4	PDD	$L_1^{[3]}(\mathbf{u}_1^{[6,7]})$	-	$\beta_3^{[4]}u_3^{[2]}+$ $L_3^{[2]}(\mathbf{u}_1^{[6,7]})$	$u_{1,3}^{[1]} =$ $L_3^{[2]}(\mathbf{u}_1^{[6,7]})$
5	PDD	$L_1^{[4]}(\mathbf{u}_1^{[8,9]})$	-	$\beta_3^{[5]}u_3^{[3]}+$ $L_3^{[3]}(\mathbf{u}_1^{[8,9]})$	$u_{1,3}^{[2]} =$ $L_3^{[3]}(\mathbf{u}_1^{[8,9]})$
6	DDD	$L_1^{[5]}(\mathbf{u}_{1,2})$	$L_2^{[4]}(\mathbf{u}_{1,2})$	$L_3^{[4]}(\mathbf{u}_{1,2})$	$u_{1,2;3} =$ $L_3^{[4]}(\mathbf{u}_{1,2})$
7	DDD	$L_1^{[6]}(\mathbf{u}_{1,3})$	$L_2^{[5]}(\mathbf{u}_{1,3})$	$L_3^{[5]}(\mathbf{u}_{1,3})$	$u_{1,3;2} =$ $L_2^{[5]}(\mathbf{u}_{1,3})$
8	DDD	$L_1^{[7]}(u_{1,2;3}, u_{1,3;2})$	$\beta_2^{[6]}u_{1,2;3}$	$\beta_3^{[6]}u_{1,3;2}$	-
9	PDD	$L_1^{[8]}(u_{1,2;3}, u_{1,3;2})$	$L_2^{[6]}(u_{2,3}, u_{1,2;3})$	$L_3^{[6]}(u_{2,3}, u_{1,3;2})$	-

4.4.2.1 $S_1^{5/3}$: phase 1

In this section, the transmission in phase 1 is described, during which the original information symbols are transmitted. Phase 1 is split into 3 transmission periods, referred to as stages, during which the order-2 symbols useful for the pairs of receivers $(\text{Rx}_2, \text{Rx}_3)$, $(\text{Rx}_1, \text{Rx}_2)$ and $(\text{Rx}_1, \text{Rx}_3)$ are generated, referred to as $(2,3)$-stage, $(1,2)$-stage and $(1,3)$-stage, respectively.

$(2,3)$-*stage:* $t = 1$. $(2,3)$-stage comprises the first channel use $t = 1$, during which the order-2 symbol $u_{2,3}$ is generated.

At $t = 1$, the symbol $u_1^{[1]}$ is transmitted using random precoding and the symbols $u_2^{[1]}$ and $u_3^{[1]}$ are transmitted using ZF to ensure that no interference is overheard by Rx$_1$. The signal transmitted at $t = 1$ is given by

$$\mathbf{x}(1) = \mathbf{c}_1(1)\, u_1^{[1]} + \mathbf{c}_{23}(1)\left(u_2^{[1]} + u_3^{[1]}\right), \tag{4.16}$$

where $\mathbf{c}_1(1) \in \mathbb{C}^{2\times 1}$ is a random precoding vector with independent entries taken from continuous distributions and $\mathbf{c}_{23}(1) \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(1)\,\mathbf{c}_{23}(1) = 0$. At $t = 1$, Rx_1 receives an interference-free signal

$$y_1(1) = \mathbf{h}_1^{\mathrm{T}}(1)\,\mathbf{c}_1(1)\,u_1^{[1]} = \beta_1^{[1]} u_1^{[1]}, \tag{4.17}$$

which allows to decode $u_1^{[1]}$. At $t = 1$, Rx_j, $j \in \{2, 3\}$, receives

$$y_j(1) = \mathbf{h}_i^{\mathrm{T}}(1)\left(\mathbf{c}_1(1)\,u_1^{[1]} + \mathbf{c}_{23}(1)\left(u_2^{[1]} + u_3^{[1]}\right)\right), \tag{4.18}$$

which can be written for Rx_2 and Rx_3 as

$$y_2(1) = \beta_2^{[1]} u_1^{[1]} + \beta_2^{[2]} u_2^{[1]} + \beta_2^{[3]} u_3^{[1]} = \beta_2^{[2]} u_2^{[1]} + L_2^{[1]}(u_1^{[1]}, u_3^{[1]}), \tag{4.19}$$

$$y_3(1) = \beta_3^{[1]} u_1^{[1]} + \beta_3^{[2]} u_2^{[1]} + \beta_3^{[3]} u_3^{[1]} = \beta_3^{[3]} u_3^{[1]} + L_3^{[1]}(u_1^{[1]}, u_2^{[1]}). \tag{4.20}$$

From the interference terms overheard by Rx_2 and Rx_3, the order-2 symbol

$$\begin{aligned} u_{2,3} &= \beta_2^{[1]}\beta_3^{[1]} u_1^{[1]} + \beta_2^{[1]}\beta_3^{[2]} u_2^{[1]} + \beta_3^{[1]}\beta_2^{[3]} u_3^{[1]} \\ &= \beta_2^{[1]}\beta_3^{[2]} u_2^{[1]} + \beta_3^{[1]} L_2^{[1]}(u_1^{[1]}, u_3^{[1]}) \\ &= \beta_3^{[1]}\beta_2^{[3]} u_3^{[1]} + \beta_2^{[1]} L_3^{[1]}(u_1^{[1]}, u_2^{[1]}) \end{aligned} \tag{4.21}$$

is generated. The delivery of $u_{2,3}$ is to allow Rx_2 and Rx_3 to cancel the interference in the received signals and decode $u_2^{[1]}$ and $u_3^{[1]}$, respectively.

$(1, 2)$-*stage:* $t = 2, 3$. $(1, 2)$-stage comprises two channel uses $t = 2, 3$, during which the order-2 symbols $u_{12}^{[1]}$ and $u_{12}^{[2]}$ are generated.

At $t = 2$, the symbol vector $\mathbf{u}_1^{[2,3]} = \begin{bmatrix} u_1^{[2]} & u_1^{[3]} \end{bmatrix}^{\mathrm{T}}$ is transmitted using random precoding and the symbol $u_2^{[2]}$ is transmitted using ZF. The signal transmitted at $t = 2$ is given by

$$\mathbf{x}(2) = \mathbf{C}_1(2)\,\mathbf{u}_1^{[2,3]} + \mathbf{c}_2(2)\,u_2^{[2]}, \tag{4.22}$$

where $\mathbf{C}_1(2) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix and $\mathbf{c}_2(2) \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(2)\,\mathbf{c}_2(2) = 0$. At $t = 2$, Rx_1 receives a useful linear combination of the elements of $\mathbf{u}_1^{[2,3]}$

$$y_1(2) = \mathbf{h}_1^{\mathrm{T}}(2)\,\mathbf{C}_1(2)\,\mathbf{u}_1^{[2,3]} = L_1^{[1]}(\mathbf{u}_1^{[2,3]}), \tag{4.23}$$

where yet another linear combination remains necessary for decoding. Rx_2 receives

$$y_2(2) = \mathbf{h}_2^{\mathrm{T}}(2)\left(\mathbf{C}_1(2)\,\mathbf{u}_1^{[2,3]} + \mathbf{c}_2(2)\,u_2^{[2]}\right) = \beta_2^{[4]} u_2^{[2]} + L_2^{[2]}(\mathbf{u}_1^{[2,3]}), \tag{4.24}$$

which is a superposition of the desired symbol $\beta_2^{[4]} u_2^{[2]}$ and the interference term $L_2^{[2]}(\mathbf{u}_1^{[2,3]})$ which is useful for Rx_1. By $u_{12}^{[1]} = L_2^{[2]}(\mathbf{u}_1^{[2,3]})$, the order-2 symbol useful for Rx_1 and Rx_2 is generated. The delivery of $u_{1,2}^{[1]}$ is to allow Rx_1 and Rx_2 to decode $\mathbf{u}_1^{[2,3]}$ and $u_2^{[2]}$, respectively.

At $t = 3$, the new symbol vector $\mathbf{u}_1^{[4,5]} = [u_1^{[4]} \quad u_1^{[5]}]^{\mathrm{T}}$ and the new symbol $u_2^{[3]}$ are transmitted. The signal transmitted at $t = 3$ is given by

$$\mathbf{x}(3) = \mathbf{C}_1(3)\,\mathbf{u}_1^{[4,5]} + \mathbf{c}_2(3)\,u_2^{[3]}, \tag{4.25}$$

where $\mathbf{C}_1(3) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix and $\mathbf{c}_2(3) \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(3)\,\mathbf{c}_2(3) = 0$. From the interference term overheard by Rx_2, the order-2 symbol $u_{1,2}^{[2]} = L_2^{[3]}(\mathbf{u}_1^{[4,5]})$ is generated.

$(1,3)$-*stage:* $t = 4, 5$. $(1,3)$-stage comprises two channel uses $t = 4, 5$ during which the order-2 symbols $u_{1,3}^{[1]}$ and $u_{1,3}^{[2]}$ are generated.

At $t = 4$, the symbol vector $\mathbf{u}_1^{[6,7]} = [u_1^{[6]} \quad u_1^{[7]}]^{\mathrm{T}}$ is transmitted using random precoding and the symbol $u_3^{[2]}$ is transmitted using ZF. The signal transmitted at $t = 4$ is given by

$$\mathbf{x}(4) = \mathbf{C}_1(4)\,\mathbf{u}_1^{[6,7]} + \mathbf{c}_3(4)\,u_3^{[2]}, \tag{4.26}$$

where $\mathbf{C}_1(4) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix and $\mathbf{c}_3(4) \in \mathbb{C}^{2\times 1}$ is a precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(4)\,\mathbf{c}_3(4) = 0$. At $t = 5$, the transmission is repeated with the new symbol vector $\mathbf{u}_1^{[8,9]} = [u_1^{[8]} \quad u_1^{[9]}]^{\mathrm{T}}$ and the symbol $u_3^{[2]}$. The signal transmitted at $t = 5$ is given by

$$\mathbf{x}(5) = \mathbf{C}_1(5)\,\mathbf{u}_1^{[8,9]} + \mathbf{c}_3(5)\,u_3^{[3]}, \tag{4.27}$$

where $\mathbf{C}_1(5) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix and $\mathbf{c}_3(5) \in \mathbb{C}^{2\times 1}$ is the precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(5)\,\mathbf{c}_3(5) = 0$.

From the interference terms received at $t = 4, 5$ by Rx_3, the order-2 symbols $u_{1,3}^{[1]} = L_3^{[2]}(\mathbf{u}_1^{[6,7]})$ and $u_{1,3}^{[2]} = L_3^{[3]}(\mathbf{u}_1^{[8,9]})$ are generated.

4.4.2.2 $S_1^{5/3}$: phase 2

In this section, the transmission in phase 2 is described, during which the order-2 symbols generated in phase 1 are delivered to the receivers which desire them. In the first two channel uses $t = 6, 7$, each having a DDD state, the order-2 symbols $u_{1,2}^{[1]}$, $u_{1,2}^{[2]}$, $u_{1,3}^{[1]}$, $u_{1,3}^{[2]}$ are transmitted, where from the interference terms overheard at the unintended receivers, two terms $u_{1,2;3}$ and $u_{1,3;2}$ useful for two receivers and known at the remaining third receiver, referred to as order-(2,1) symbols, are generated. In the remaining two channel uses $t = 8$ having DDD state and $t = 9$ having PDD state, the freshly generated order-(2,1) symbols $u_{1,2;3}$ and $u_{1,3;2}$ and the order-2 symbol $u_{2,3}$ are delivered to the receivers which desire them.

At $t = 6$, the order-2 symbol vector $\mathbf{u}_{1,2} = \left[u_{1,2}^{[1]} \quad u_{1,2}^{[2]}\right]^{\mathrm{T}}$ is transmitted to Rx_1 and Rx_2 using random precoding. The signal transmitted at $t = 6$ is given by

$$\mathbf{x}(6) = \mathbf{C}_{1,2}(6)\,\mathbf{u}_{1,2}, \tag{4.28}$$

where $\mathbf{C}_{1,2}(6) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix. The signal received at $t = 6$ by Rx_i, $i \in \{1, 2, 3\}$, is

$$y_i(6) = \mathbf{h}_i^{\mathrm{T}}(6)\,\mathbf{C}_{1,2}(6)\,\mathbf{u}_{1,2}, \tag{4.29}$$

in which Rx_1 and Rx_2 receive useful linear combinations of the elements of $\mathbf{u}_{1,2}$ and Rx_3 receives an interference term. From the interference term received by Rx_3, an order-(2,1) symbol $u_{1,2;3} = y_3(6) = L_3^{[4]}(\mathbf{u}_{1,2})$ useful for Rx_1 and Rx_2 and known at Rx_3 is generated. The delivery of $u_{1,2;3}$ is to allow Rx_1 and Rx_2 to decode $\mathbf{u}_{1,2}$.

At $t = 7$, the order-2 symbol vector $\mathbf{u}_{1,3} = \left[u_{1,3}^{[1]} \quad u_{1,3}^{[2]}\right]^{\mathrm{T}}$ is transmitted to Rx_1 and Rx_3 using random precoding. The signal transmitted at $t = 7$ is given by

$$\mathbf{x}(7) = \mathbf{C}_{1,3}(7)\,\mathbf{u}_{1,3}, \tag{4.30}$$

where $\mathbf{C}_{1,3}(7) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix. From the interference term overheard by Rx_2, an order-(2,1) symbol $u_{1,3;2} = L_2^{[5]}(\mathbf{u}_{1,3})$ useful for Rx_1 and Rx_3 and known at Rx_2 is generated.

At $t = 8, 9$, the freshly generated order-(2,1) symbols $u_{1,2;3}$ and $u_{1,3;2}$ are transmitted using random precoding and the remaining order-2 symbol $u_{2,3}$ is transmitted using ZF. The signals transmitted at $t = 8, 9$ are given by

$$\mathbf{x}(8) = \mathbf{c}_{1,2;3}(8)\,u_{1,2;3} + \mathbf{c}_{1,3;2}(8)\,u_{1,3;2}, \tag{4.31}$$

$$\mathbf{x}(9) = \mathbf{c}_{1,2;3}(9)\,u_{1,2;3} + \mathbf{c}_{1,3;2}(9)\,u_{1,3;2} + \mathbf{c}_{2,3}(9)\,u_{2,3}, \tag{4.32}$$

where $\mathbf{c}_{1,2;3}(8), \mathbf{c}_{1,3;2}(8), \mathbf{c}_{1,2;3}(9), \mathbf{c}_{1,3;2}(9) \in \mathbb{C}^{2\times 1}$ are random precoding vectors and $\mathbf{c}_{2,3}(9) \in \mathbb{C}^{2\times 1}$ is the precoding vector satisfying $\mathbf{h}_1^{\mathrm{T}}(9)\,\mathbf{c}_{23}(9) = 0$. At $t = 8, 9$, Rx_1 receives

$$y_1(t) = \mathbf{h}_1^{\mathrm{T}}(t)\left(\mathbf{c}_{1,2;3}(t)\,u_{1,2;3} + \mathbf{c}_{1,3;2}(t)\,u_{1,3;2}\right) \tag{4.33}$$

from which $u_{1,2;3}$ and $u_{1,3;2}$ are decoded. The signals received by Rx_2 and Rx_3 are given by

$$y_j(8) = \mathbf{h}_j^{\mathrm{T}}(8)\left(\mathbf{c}_{1,2;3}(8)\,u_{1,2;3} + \mathbf{c}_{1,3;2}(8)\,u_{1,3;2}\right), \tag{4.34}$$

$$y_j(9) = \mathbf{h}_j^{\mathrm{T}}(9)\left(\mathbf{c}_{1,2;3}(9)\,u_{1,2;3} + \mathbf{c}_{1,3;2}(9)\,u_{1,3;2} + \mathbf{c}_{2,3}(9)\,u_{2,3}\right), \tag{4.35}$$

Table 4.4: Summary of the constituent encoding schemes achieving $\frac{3}{2}$ DoF.

CS	State fractions	DoF tuples	Achievability
$S_1^{3/2}$	$\lambda_{\text{DDD}} = 1$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	[MAT12]
$S_2^{3/2}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$ and $\left(1, 0, \frac{1}{2}\right)$	[TJSSP13]
$S_3^{3/2}$	$(\lambda_{\text{PDP}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$	[TJSSP13]
$S_4^{3/2}$	$(\lambda_{\text{PPD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, 0, \frac{1}{2}\right)$ and $\left(0, 1, \frac{1}{2}\right)$	[TJSSP13]
$S_5^{3/2}$	$(\lambda_{\text{PPD}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(0, 1, \frac{1}{2}\right)$	[TJSSP13]

$j \in \{2, 3\}$, from which Rx_2 and Rx_3 decode the desired symbols after subtracting the known order-(2,1) symbols.

4.4.3 Schemes Achieving $\frac{3}{2}$ DoF

In this section, CSs achieving $\frac{3}{2}$ DoF are described.

$S_1^{3/2}$ is the transmission scheme performing encoding over DDD state, for which we refer to the known transmission scheme in [MAT12]. $S_2^{3/2}$, $S_3^{3/2}$, $S_4^{3/2}$ and $S_5^{3/2}$ are the transmission schemes for achieving non-optimal DoF corner points which perform joint encoding over the CSIT state pairs (PDD, DDD), (PPD, DDD), (PDP, DDD) and (PPD, PDD) respectively. For $S_2^{3/2}$, $S_3^{3/2}$, $S_4^{3/2}$, and $S_5^{3/2}$, we refer to the transmission scheme for the 2-user MISO BC given in [TJSSP13], which jointly encodes over PD and DD states achieving the DoF tuple $(d_1, d_2) = \left(1, \frac{1}{2}\right)$. The details of the CSs achieving $\frac{3}{2}$ DoF are given in Table 4.4.

4.4.4 Scheme Achieving $\frac{12}{7}$ DoF

In this section, the third newly proposed transmission scheme $S^{12/7}$ is described, in which joint encoding over PDD, DPD and DDD states is performed. $S^{12/7}$ achieves the DoF tuple $(d_1, d_2, d_3) = \left(1, \frac{3}{7}, \frac{2}{7}\right)$ for the CSIT state fractions $(\lambda_{\text{PDD}}, \lambda_{\text{DPD}}, \lambda_{\text{DDD}}) = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{7}\right)$.

In $S^{12/7}$, joint encoding over 5 PDD, 1 DPD and 1 DDD states is performed. During the transmission, seven symbols $\{u_1^{[k]}\}_{k=1}^7$ are delivered to Rx_1, three symbols $\{u_2^{[k]}\}_{k=1}^3$ are delivered to Rx_2 and two symbols $\{u_3^{[k]}\}_{k=1}^2$, are delivered to Rx_3.

Table 4.5: Summary of phase 2 of the scheme $S^{12/7}$

t	State	Rx_1	Rx_2	Rx_3	Generated Symbol
5	DDD	$L_1^{[1]}(\mathbf{u}_{1,2})$	$L_2^{[1]}(\mathbf{u}_{1,2})$	$L_3^{[1]}(\mathbf{u}_{1,2})$	$u_{1,2;3} = L_3^{[1]}(\mathbf{u}_{1,2})$
6	PDD	$\beta_1^{[1]} u_{1,2;3}$	$L_2^{[2]}(u_{2,3}, u_{1,2;3})$	$\beta_3^{[1]} u_{2,3}$	-
7	DPD	$L_1^{[2]}(u_{1,3}, u_{1,2;3})$	$\beta_2^{[1]} u_{1,2;3}$	$\beta_3^{[2]} u_{1,3}$	-

The transmission is split into two phases. Phase 1 comprises the first four channel uses $t = 1, 2, 3, 4$, each having a PDD state, during which the original information symbols are transmitted. From the interference terms overheard in phase 1, four order-2 symbols $u_{1,2}^{[1]}$, $u_{1,2}^{[2]}$, $u_{1,3}$ and $u_{2,3}$ are generated. The transmission in phase 1 is identical to the transmission in phase 1 of the scheme S_5^2 for the channel uses $t = 1, 2, 3, 4$, hence we omit further details here.

The delivery of the generated order-2 symbols is performed in phase 2 which comprises the remaining three channel uses $t = 5, 6, 7$. At $t = 5$ having DDD state, the order-2 symbols $u_{1,2}^{[1]}$ and $u_{1,2}^{[2]}$ are transmitted to Rx_1 and Rx_2, where from the interference overheard by Rx_3, an order-(2,1) symbol $u_{1,2;3}$ is generated. In the remaining channel uses $t = 6$ having DPD state and $t = 7$ having PDD state, the order-2 symbols $u_{1,3}$ and $u_{2,3}$ along with the order-(2,1) symbol $u_{1,2;3}$ are delivered to the receivers which desire them. The summary of the transmission in phase 2 is given in Table 4.5, where overheard interference terms are marked by red. The detailed description of the transmission in phase 2 is provided below.

At $t = 5$, the order-2 symbol vector $\mathbf{u}_{1,2} = [u_{1,2}^{[1]} \quad u_{1,2}^{[2]}]^{\mathrm{T}}$ is transmitted using random precoding. The signal transmitted at $t = 5$ is given by

$$\mathbf{x}(5) = \mathbf{C}_{1,2}(5)\, \mathbf{u}_{1,2}, \tag{4.36}$$

where $\mathbf{C}_{1,2}(5) \in \mathbb{C}^{2\times 2}$ is a random precoding matrix. From the interference term received by Rx_3, an order-(2,1) symbol $u_{1,2;3} = y_3(5) = L_3^{[3]}(\mathbf{u}_{1,2})$ is generated.

At $t = 6, 7$, the order-2 symbols $u_{1,3}$ and $u_{2,3}$ are transmitted using ZF and the order-(2,1) symbol $u_{1,2;3}$ is transmitted using random precoding. The signals transmitted at $t = 6, 7$ are given by

$$\mathbf{x}(6) = \mathbf{c}_{1,3}(6)\, u_{1,3} + \mathbf{c}_{1,2;3}(6)\, u_{1,2;3}, \tag{4.37}$$

$$\mathbf{x}(7) = \mathbf{c}_{2,3}(7)\, u_{2,3} + \mathbf{c}_{1,2;3}(7)\, u_{1,2;3}, \tag{4.38}$$

where $\mathbf{c}_{1,2;3}(6), \mathbf{c}_{1,2;3}(7) \in \mathbb{C}^{2\times 1}$ are random precoding vectors and $\mathbf{c}_{1,3}(6), \mathbf{c}_{2,3}(6) \in \mathbb{C}^{2\times 1}$ are precoding vectors satisfying $\mathbf{h}_2^{\mathrm{T}}(6)\,\mathbf{c}_{1,3}(6) = \mathbf{h}_1^{\mathrm{T}}(7)\,\mathbf{c}_{2,3}(7) = 0$.

The signals received by Rx_1 at $t = 6, 7$ are given by

$$y_1(6) = \mathbf{h}_1^{\mathrm{T}}(6)\left(\mathbf{c}_{1,3}(6)\,u_{1,3} + \mathbf{c}_{1,2;3}(6)\,u_{1,2;3}\right), \tag{4.39}$$

$$y_1(7) = \mathbf{h}_1^{\mathrm{T}}(7)\,\mathbf{c}_{1,2;3}(7)\,u_{1,2;3}, \tag{4.40}$$

from which Rx_1 decodes $u_{1,3}$ and $u_{1,2;3}$. The signals received by Rx_2 are

$$y_2(6) = \mathbf{h}_2^{\mathrm{T}}(6)\,\mathbf{c}_{1,2;3}(6)\,u_{1,2;3}, \tag{4.41}$$

$$y_2(7) = \mathbf{h}_2^{\mathrm{T}}(7)\left(\mathbf{c}_{2,3}(7)\,u_{2,3} + \mathbf{c}_{1,2;3}(7)\,u_{1,2;3}\right), \tag{4.42}$$

from which Rx_2 decodes $u_{1,2}$ and $u_{1,2;3}$. The signals received by Rx_3 at $t = 6, 7$ are

$$y_3(6) = \mathbf{h}_3^{\mathrm{T}}(6)\left(\mathbf{c}_{1,3}(6)\,u_{1,3} + \mathbf{c}_{1,2;3}(6)\,u_{1,2;3}\right), \tag{4.43}$$

$$y_3(7) = \mathbf{h}_3^{\mathrm{T}}(7)\left(\mathbf{c}_{2,3}(7)\,u_{2,3} + \mathbf{c}_{1,2;3}(7)\,u_{1,2;3}\right), \tag{4.44}$$

from which Rx_3 decodes $u_{1,2}$ and $u_{1,3}$ after subtracting the known order-(2,1) symbol $u_{1,2;3}$.

4.4.5 Scheme Achieving $\frac{33}{19}$ DoF

In this section, the fourth newly proposed transmission scheme $S^{33/19}$ is described, in which joint encoding over PDD, DPD and DDP states is performed. $S^{33/19}$ achieves the DoF tuple $(d_1, d_2, d_3) = \left(1, \frac{7}{19}, \frac{7}{19}\right)$ for the CSIT state fractions $(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDP}}) = \left(\frac{15}{19}, \frac{2}{19}, \frac{2}{19}\right)$.

In $S^{33/19}$, joint encoding over 15 PDD, 2 DPD and 2 DDP states is performed. During the transmission, 19 symbols $\{u_1^{[k]}\}_{k=1}^{19}$ are delivered to Rx_1, seven symbols $\{u_2^{[k]}\}_{k=1}^{7}$ are delivered to Rx_2 and seven symbols $\{u_3^{[k]}\}_{k=1}^{7}$ are delivered to Rx_3.

The transmission is split into two phases. Phase 1 comprises the first 11 channel uses $t = 1, \ldots, 11$, each having a PDD state, during which the original information symbols are transmitted. From the interference terms overheard in phase 1, 11 order-2 symbols $\{u_{2,3}^{[k]}\}_{k=1}^{3}$, $\{u_{1,2}^{[k]}\}_{k=1}^{4}$ and $\{u_{1,3}^{[k]}\}_{k=1}^{4}$ are generated. The transmission of the original symbols in phase 1 and the order-2 symbol generation closely follow the scheme $S_1^{5/3}$, hence we omit the further details here.

The transmission of the generated order-2 symbols is performed in phase 2 which comprises the remaining channel uses $t = 12, \ldots, 19$. At $t = 12, 13$, each having PDD

Table 4.6: Summary of phase 2 of the scheme $S^{33/19}$

t	State	Rx$_1$	Rx$_2$	Rx$_3$	Generated Symbol
12	PDD	$L_1^{[1]}(\mathbf{u}_{1,2})$	$L_2^{[1]}(\mathbf{u}_{1,2}, u_{2,3})$	$\beta_3^{[1]} u_{2,3} + L_3^{[1]}(\mathbf{u}_{1,2})$	$u_{1,2,3}^{[1]} = L_3^{[1]}(\mathbf{u}_{1,2})$
13	PDD	$L_1^{[2]}(\mathbf{u}_{1,3})$	$\beta_2^{[1]} u_{2,3} + L_2^{[2]}(\mathbf{u}_{1,3})$	$L_3^{[2]}(\mathbf{u}_{1,3}, u_{2,3})$	$u_{1,2,3}^{[2]} = L_2^{[2]}(\mathbf{u}_{1,3})$
14	PDD	$\beta_1^{[1]} u_{1,2,3}^{[1]}$	$L_2^{[3]}(u_{2,3}^{[2]}, u_{1,2,3}^{[1]})$	$L_3^{[3]}(u_{2,3}^{[2]}, u_{1,2,3}^{[1]})$	-
15	DPD	$L_1^{[3]}(u_{1,3}^{[3]}, u_{1,2,3}^{[1]})$	$\beta_2^{[2]} u_{1,2,3}^{[1]}$	$L_3^{[4]}(u_{1,3}^{[3]}, u_{1,2,3}^{[1]})$	-
16	DDP	$L_1^{[4]}(u_{1,2}^{[3]}, u_{1,2,3}^{[1]})$	$L_2^{[4]}(u_{1,2}^{[3]}, u_{1,2,3}^{[1]})$	$\beta_3^{[2]} u_{1,2,3}^{[1]}$	-
17	PDD	$\beta_1^{[2]} u_{1,2,3}^{[2]}$	$L_2^{[5]}(u_{2,3}^{[3]}, u_{1,2,3}^{[2]})$	$L_3^{[5]}(u_{2,3}^{[3]}, u_{1,2,3}^{[2]})$	-
18	DPD	$L_1^{[5]}(u_{1,3}^{[4]}, u_{1,2,3}^{[2]})$	$\beta_2^{[3]} u_{1,2,3}^{[1]}$	$L_3^{[6]}(u_{1,3}^{[4]}, u_{1,2,3}^{[2]})$	-
19	DDP	$L_1^{[6]}(u_{1,2}^{[4]}, u_{1,2,3}^{[2]})$	$L_2^{[6]}(u_{1,2}^{[4]}, u_{1,2,3}^{[2]})$	$\beta_3^{[3]} u_{1,2,3}^{[2]}$	-

state, the order-2 symbols $\{u_{1,2}^{[k]}\}_{k=1}^{2}$, $\{u_{1,3}^{[k]}\}_{k=1}^{2}$ and $u_{2,3}^{[1]}$ are transmitted, where from the interference terms overheard by Rx$_2$ and Rx$_3$, terms $\{u_{1,2,3}^{[k]}\}_{k=1}^{2}$ useful for all receivers, referred to in the following as order-3 symbols, are generated. In the remaining channel uses $t = 14, \dots, 19$ comprised of 2 PDD states, 2 DPD states and 2 DDP states, the freshly generated order-3 symbols $\{u_{1,2,3}^{[k]}\}_{k=1}^{2}$ and the remaining order-2 symbols $\{u_{1,2}^{[k]}\}_{k=3}^{4}$, $\{u_{1,3}^{[k]}\}_{k=3}^{4}$ and $\{u_{2,3}^{[k]}\}_{k=2}^{3}$ are delivered to the receivers which desire them. The summary of the transmission in phase 2 is given in Table 4.6, where overheard interference terms are marked by red. The detailed description of the transmission in phase 2 is provided below.

At $t = 12, 13$, the order-2 symbol vectors $\mathbf{u}_{1,2} = \left[u_{1,2}^{[1]} \quad u_{1,2}^{[2]}\right]^{\mathrm{T}}$ and $\mathbf{u}_{1,3} = \left[u_{1,3}^{[1]} \quad u_{1,2}^{[2]}\right]^{\mathrm{T}}$ are transmitted using random precoding and the order-2 symbol $u_{2,3}^{[1]}$ is transmitted using ZF. The signals transmitted at $t = 12, 13$ are given by

$$\mathbf{x}(12) = \mathbf{C}_{1,2}(12)\,\mathbf{u}_{1,2} + \mathbf{c}_{2,3}(12)\,u_{2,3}, \tag{4.45}$$

$$\mathbf{x}(13) = \mathbf{C}_{1,3}(13)\,\mathbf{u}_{1,3} + \mathbf{c}_{2,3}(13)\,u_{2,3}, \tag{4.46}$$

where $\mathbf{C}_{1,2}(12), \mathbf{C}_{1,2}(13) \in \mathbb{C}^{2\times 2}$ are random precoding matrices and $\mathbf{c}_{2,3}(12), \mathbf{c}_{2,3}(13) \in \mathbb{C}^{2\times 1}$ are the precoding vectors satisfying $\mathbf{h}_1^{\mathrm{T}}(12)\,\mathbf{c}_{2,3}(12) =$

$\mathbf{h}_1^{\mathrm{T}}(13)\,\mathbf{c}_{2,3}(13) = 0$. The signals received by Rx_1 at $t = 12, 13$ are given by

$$y_1(12) = \mathbf{h}_1^{\mathrm{T}}(12)\,\mathbf{C}_{1,2}(12)\,\mathbf{u}_{1,2}, \tag{4.47}$$
$$y_1(13) = \mathbf{h}_1^{\mathrm{T}}(13)\,\mathbf{C}_{1,3}(13)\,\mathbf{u}_{1,3}, \tag{4.48}$$

from which Rx_1 obtains two useful linear combinations $\mathbf{u}_{1,2}$ and $\mathbf{u}_{1,3}$. The signals received by Rx_j at $t = 12, 13$, $j \in \{2, 3\}$, are given by

$$y_j(12) = \mathbf{h}_2^{\mathrm{T}}(12)\left(\mathbf{C}_{1,2}(12)\,\mathbf{u}_{1,2} + \mathbf{c}_{2,3}(12)\,u_{2,3}\right), \tag{4.49}$$
$$y_j(13) = \mathbf{h}_2^{\mathrm{T}}(13)\left(\mathbf{C}_{1,3}(13)\,\mathbf{u}_{1,3} + \mathbf{c}_{2,3}(13)\,u_{2,3}\right), \tag{4.50}$$

where Rx_2 and Rx_3 obtain useful linear combinations at $t = 12$ and $t = 13$, respectively. At $t = 12$, Rx_3 receives a superposition of desired symbol and interference term $\beta_3^{[1]} u_{2,3} + L_3^{[1]}(\mathbf{u}_{1,2})$. From $L_3^{[1]}(\mathbf{u}_{1,2})$ an order-3 symbol $u_{1,2,3}^{[1]} = L_3^{[1]}(\mathbf{u}_{1,2})$ is generated. At $t = 13$, Rx_3 receives a superposition of desired symbol and interference term $\beta_2^{[1]} u_{2,3} + L_2^{[2]}(\mathbf{u}_{1,3})$. From $L_2^{[2]}(\mathbf{u}_{1,3})$, an order-3 symbol $u_{1,2,3}^{[2]} = L_2^{[2]}(\mathbf{u}_{1,3})$ is generated. The delivery of $u_{1,2,3}^{[1]}$ and $u_{1,2,3}^{[2]}$ is to allow each receiver to decode the desired order-2 symbols.

In the next three channel uses $t = 14, 15, 16$, the freshly generated order-3 symbol $u_{1,2,3}^{[1]}$ and the order-2 symbols $u_{2,3}^{[2]}$, $u_{1,2}^{[3]}$ and $u_{1,3}^{[3]}$ are delivered to the receivers which desire them. The signals transmitted at $t = 14, 15, 16$ are given by

$$\mathbf{x}(14) = \mathbf{c}_{2,3}(14)\,u_{2,3}^{[2]} + \mathbf{c}_{1,2,3}(14)\,u_{1,2,3}^{[1]}, \tag{4.51}$$
$$\mathbf{x}(15) = \mathbf{c}_{1,3}(15)\,u_{1,3}^{[3]} + \mathbf{c}_{1,2,3}(15)\,u_{1,2,3}^{[1]}, \tag{4.52}$$
$$\mathbf{x}(16) = \mathbf{c}_{1,2}(16)\,u_{1,2}^{[3]} + \mathbf{c}_{1,2,3}(16)\,u_{1,2,3}^{[1]}, \tag{4.53}$$

where $\mathbf{c}_{1,2,3}(t) \in \mathbb{C}^{2\times 1}$ is a random precoding vector and $\mathbf{c}_{2,3}(14), \mathbf{c}_{1,3}(15), \mathbf{c}_{1,2}(16) \in \mathbb{C}^{2\times 1}$ are the precoding vectors satisfying $\mathbf{h}_1^{\mathrm{T}}(14)\,\mathbf{c}_{2,3}(14) = \mathbf{h}_2^{\mathrm{T}}(15)\,\mathbf{c}_{1,3}(15) = 0$. The signals received at $t = 14, 15, 16$ by Rx_1 are given by

$$\mathbf{y}_1(14) = \mathbf{h}_1^{\mathrm{T}}(14)\,\mathbf{c}_{1,2,3}(14)\,u_{1,2,3}^{[1]}, \tag{4.54}$$
$$\mathbf{y}_1(15) = \mathbf{h}_1^{\mathrm{T}}(15)\left(\mathbf{c}_{1,3}(15)\,u_{1,3}^{[3]} + \mathbf{c}_{1,2,3}(15)\,u_{1,2,3}^{[1]}\right), \tag{4.55}$$
$$\mathbf{y}_1(16) = \mathbf{h}_1^{\mathrm{T}}(16)\left(\mathbf{c}_{1,2}(16)\,u_{1,2}^{[3]} + \mathbf{c}_{1,2,3}(16)\,u_{1,2,3}^{[1]}\right), \tag{4.56}$$

from which Rx_1 decodes $u_{1,2,3}^{[1]}$, $u_{1,2}^{[3]}$ and $u_{1,3}^{[3]}$. Due to symmetry, we omit the description of the decoding by other receivers. At $t = 17, 18, 19$ the remaining order-3 symbol $u_{1,2,3}^{[2]}$ and the remaining order-2 symbols $u_{2,3}^{[3]}$, $u_{1,2}^{[4]}$ and $u_{1,3}^{[4]}$ are delivered to the receivers which desire them. The transmission at $t = 17, 18, 19$ follows the one at $t = 14, 15, 16$, hence we omit further details here.

4.5 Achievability for Theorem 3

In this section, the achievability of the DoF outer bound (4.4) is shown, completing thus the proof of Theorem 3. First, we show the achievability of the optimal DoF region corner points. Then, we show the achievability of the remaining non-optimal DoF region corner points.

4.5.1 Achievability of the Optimal DoF Corner Points

In this section, we show the achievability of the optimal DoF region corner points. Below, we provide the CS assignment procedure which achieves the optimal DoF region corner points for each of the regions specified in Corollary 2 of Section 4.3.

The encoding over PPD and PDP states is performed independently using the schemes S_1^2 and S_2^2, respectively, where the CS fractions are given by $\lambda_{S_1^2} = \lambda_{\text{PPD}}$ and $\lambda_{S_2^2} = \lambda_{\text{PDP}}$. The encoding over the remaining PPP, PDD and DDD states is performed jointly, with the following assignment. Initially, the scheme S_5^2 is applied for joint encoding over PPP and PDD states. Depending on whether λ_{PDD} is greater or smaller than λ_{PPP}, the following two cases are distinguished.

1) Case A: $\lambda_{\text{PDD}} \geq \lambda_{\text{PPP}}$. In this case, PPP state can be fully exhausted using S_5^2 with the CS fraction $\lambda_{S_5^2} = 2\lambda_{\text{PPP}}$. The remaining PDD state fraction $\lambda^*_{\text{PDD}} = \lambda_{\text{PDD}} - \lambda_{\text{PPP}}$ is alternated with DDD state using the scheme $S_1^{5/3}$. Depending on whether $2\lambda_{\text{DDD}}$ is greater or smaller than λ^*_{PDD}, the following two sub-cases are distinguished.

A.1. $2\lambda_{\text{DDD}} \geq \lambda^*_{\text{PDD}}$: The remaining fraction of PDD state can be fully exhausted using $S_1^{5/3}$ with the CS fraction $\lambda_{S_1^{5/3}} = \frac{3}{2}\lambda^*_{\text{PDD}}$. Over the remaining DDD state fraction, encoding using $S_1^{3/2}$ is performed with the CS fraction $\lambda_{S_1^{3/2}} = \lambda_{\text{DDD}} - \frac{\lambda^*_{\text{PDD}}}{2}$.

A.2. $2\lambda_{\text{DDD}} < \lambda^*_{\text{PDD}}$: DDD state can be fully exhausted using $S_1^{5/3}$. Over all available PDD and DDD states, joint encoding is performed with the CS fraction $\lambda_{S_1^{5/3}} = \lambda^*_{\text{PDD}} + \lambda_{\text{DDD}}$.

2) Case B: $\lambda_{\text{PDD}} < \lambda_{\text{PPP}}$. In this case, PDD state can be fully exhausted using S_5^2 with the CS fraction $\lambda_{S_5^2} = 2\lambda_{\text{PDD}}$. The remaining PPP state fraction $\lambda^*_{\text{PPP}} = \lambda_{\text{PPP}} - \lambda_{\text{PDD}}$ is alternated with DDD state using the scheme S_8^2. Depending on whether $2\lambda_{\text{DDD}}$ is greater or smaller than λ^*_{PPP}, the following two sub-cases are distinguished.

B.1. $2\lambda_{\text{DDD}} \geq \lambda^*_{\text{PPP}}$: The remaining PPP state fraction can be fully exhausted using S_8^2 with the CS fraction $\lambda_{S_8^2} = \frac{3}{2}\lambda^*_{\text{PPP}}$. Over the remaining DDD state fraction, encoding using $S_1^{3/2}$ is performed with the CS fraction $\lambda_{S_1^{3/2}} = \lambda_{\text{DDD}} - \frac{\lambda^*_{\text{PPP}}}{2}$.

B.2. $2\lambda_{\text{DDD}} < \lambda^*_{\text{PPP}}$: DDD state can be fully exhausted using $\lambda_{S_8^2}$ with the CS fraction $\lambda_{S_8^2} = 3\lambda_{\text{DDD}}$. Over the remaining PPP state fraction, encoding using S_4^2 is performed with the CS fraction $\lambda_{S_4^2} = \lambda^*_{\text{PPP}} - 2\lambda_{\text{DDD}}$.

The summary of the CS assignment for cases A.1, A.2, B.1 and B.2 is given in Tables 4.7, 4.8, 4.9 and 4.10, respectively.

Equivalence to (4.5)*:* In the restricted alternating CSIT setting, $2\lambda_{\text{DDD}} \geq \lambda^*_{\text{PDD}}$ is equivalent to $3\lambda_1 - \lambda_2 - \lambda_3 \leq 2$ and $2\lambda_{\text{DDD}} \geq \lambda^*_{\text{PPP}}$ is equivalent to $\lambda_1 + \lambda_2 + \lambda_3 \leq 2$. Hence, Cases A.1 and B.1 correspond to Region I with $\lambda_{\text{PDD}} \geq \lambda_{\text{PPP}}$ and $\lambda_{\text{PDD}} < \lambda_{\text{PPP}}$, respectively. Consecutively, Cases A.2 and B.2 correspond to Regions II and III, respectively. Using the relationship between the marginal and joint CSIT state probabilities

$$\lambda_1 = \lambda_{\text{PPP}} + \lambda_{\text{PPD}} + \lambda_{\text{PDP}} + \lambda_{\text{PDD}}, \tag{4.57}$$

$$\lambda_2 = \lambda_{\text{PPP}} + \lambda_{\text{PPD}}, \tag{4.58}$$

$$\lambda_3 = \lambda_{\text{PPP}} + \lambda_{\text{PDP}}, \tag{4.59}$$

yields the required DoF tuples.

4.5.2 Achievability of the Non-Optimal DoF Corner Points

In this section, we show the achievability of the remaining non-optimal DoF region corner points. First we give a summary of the DoF region corner points to be achieved and then show their achievability.

4.5.2.1 Summary of the DoF Corner Points

In this section, we give a summary of the non-optimal DoF region corner points to be achieved. DoF tuples having $d_i = 0$ for Rx_i, $i \in \{1, 2, 3\}$, can be achieved by referring to the transmission schemes for the 2-user MISO BC in [TJSSP13] and [CE13]. The additional non-optimal DoF region corner points are given below.

Regions I and III:

Table 4.7: Case A.1: achieving $\mathbf{A}_{\mathrm{I}} = \left(\frac{1}{2} + \frac{3\lambda_1-\lambda_2-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_2-\lambda_1-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_3-\lambda_1-\lambda_2}{4}\right)$ for $\lambda_{\mathrm{PDD}} \geq \lambda_{\mathrm{PPP}}$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PPP}}$
$S_1^{5/3}$	$(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	$\frac{3}{2}\left(\lambda_{\mathrm{PDD}} - \lambda_{\mathrm{PPP}}\right)$
$S_1^{3/2}$	$\lambda_{\mathrm{DDD}} = 1$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	$\lambda_{\mathrm{DDD}} - \frac{\lambda_{\mathrm{PDD}}-\lambda_{\mathrm{PPP}}}{2}$

Table 4.8: Case A.2: achieving $\mathbf{A}_{\mathrm{II}} = \left(1, \frac{1}{3} + \frac{2\lambda_2-\lambda_3}{3}, \frac{1}{3} + \frac{2\lambda_3-\lambda_2}{3}\right)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PPP}}$
$S_1^{5/3}$	$(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	$\lambda_{\mathrm{PDD}} - \lambda_{\mathrm{PPP}} + \lambda_{\mathrm{DDD}}$

Table 4.9: Case B.1: achieving $\mathbf{A}_{\mathrm{I}} = \left(\frac{1}{2} + \frac{3\lambda_1-\lambda_2-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_2-\lambda_1-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_3-\lambda_1-\lambda_2}{4}\right)$ for $\lambda_{\mathrm{PDD}} < \lambda_{\mathrm{PPP}}$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1, 1, 0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1, 0, 1)$	λ_{PDP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PDD}}$
S_8^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	$\frac{3}{2}\left(\lambda_{\mathrm{PPP}} - \lambda_{\mathrm{PDD}}\right)$
$S_1^{3/2}$	$\lambda_{\mathrm{DDD}} = 1$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\lambda_{\mathrm{DDD}} - \frac{\lambda_{\mathrm{PPP}}-\lambda_{\mathrm{PDD}}}{2}$

Table 4.10: Case B.2: achieving $\mathbf{A}_{\mathrm{III}}^{[1]} = (\lambda_1, \lambda_2, 2-\lambda_1-\lambda_2)$, $\mathbf{A}_{\mathrm{III}}^{[2]} = (\lambda_1, 2-\lambda_1-\lambda_3, \lambda_3)$ and $\mathbf{A}_{\mathrm{III}}^{[3]} = (2-\lambda_2-\lambda_3, \lambda_2, \lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1, 1, 0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PPD}} = 1$	$(1, 1, 0)$	λ_{PPD}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PDD}}$
S_8^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	$3\lambda_{\mathrm{DDD}}$
S_4^2	$\lambda_{\mathrm{PPP}} = 1$	$(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$	$\lambda_{\mathrm{PPP}} - \lambda_{\mathrm{PDD}} - 2\lambda_{\mathrm{DDD}}$

1. $2\lambda_1 - \lambda_3 < 1$.

   There are no additional corner points.

2. $2\lambda_1 - \lambda_3 \geq 1$, $2\lambda_1 - \lambda_2 < 1$.

   In this case, there is an additional corner point $\mathbf{B}_{13} = (1, 2\lambda_1 - \lambda_3 - 1, 1 - \lambda_1 + \lambda_3)$ with $d_\Sigma = 1 + \lambda_1$. Furthermore, the following two sub-cases are distinguished

   2.1. $2\lambda_2 - \lambda_3 < 1$.
   There are no additional corner points.

   2.2. $2\lambda_2 - \lambda_3 \geq 1$.
   There is an additional corner point $\mathbf{B}_{23} = (2\lambda_2 - \lambda_3 - 1, 1, 1 - \lambda_2 + \lambda_3)$ with $d_\Sigma = 1 + \lambda_2$.

3. $2\lambda_1 - \lambda_2 \geq 1$.

   In this case, there are two additional corner points $\mathbf{B}_{12} = (1, 1 - \lambda_1 + \lambda_2, 2\lambda_1 - \lambda_2 - 1,)$ and $\mathbf{B}_{13} = (1, 2\lambda_1 - \lambda_3 - 1, 1 - \lambda_1 + \lambda_3)$ with $d_\Sigma = 1 + \lambda_1$. Furthermore, the following two sub-cases are distinguished.

   3.1. $2\lambda_2 - \lambda_3 < 1$.
   There are no additional corner points.

   3.2. $2\lambda_2 - \lambda_3 \geq 1$.
   There is an additional corner point $\mathbf{B}_{23} = (2\lambda_2 - \lambda_3 - 1, 1, 1 - \lambda_2 + \lambda_3)$ with $d_\Sigma = 1 + \lambda_2$.

Region II: $\mathbf{B}_{12}$ and $\mathbf{B}_{13}$ lie on A_1. Since in Region II, A_1 is inactive, $\mathbf{B}_{12}$ and $\mathbf{B}_{13}$ are inactive as well. This results in only the following two cases.

1. $2\lambda_2 - \lambda_3 < 1$.

   There are no additional corner points.

2. $2\lambda_2 - \lambda_3 \geq 1$.

   There is an additional corner point $\mathbf{B}_{23} = (2\lambda_2 - \lambda_3 - 1, 1, 1 - \lambda_2 + \lambda_3)$ with $d_\Sigma = 1 + \lambda_2$.

The additional non-optimal DoF region corner points are depicted in Fig. 4.2, 4.3 and 4.4.

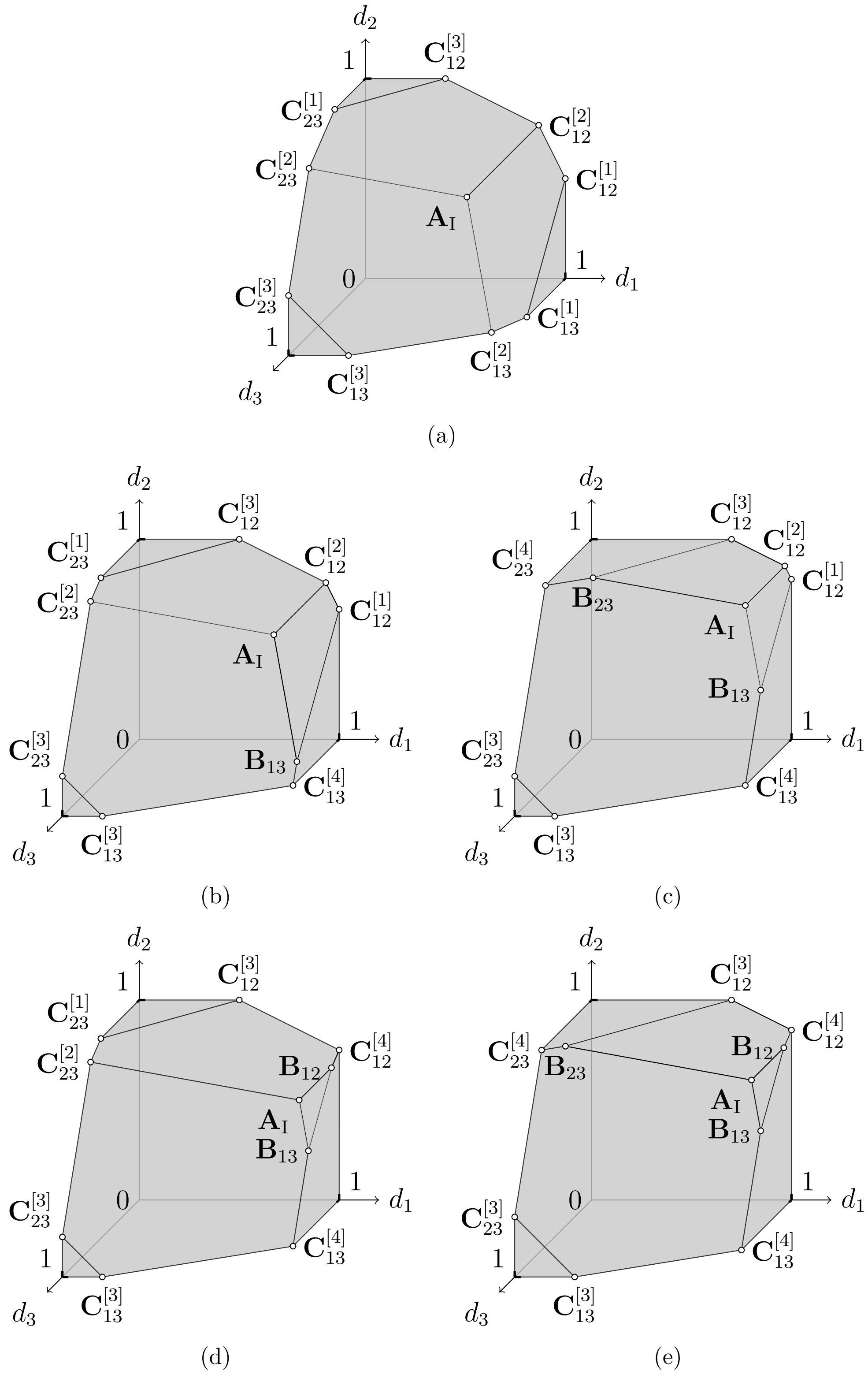

Figure 4.2: Shapes of the DoF regions for Region I: (a) Case 1, (b) Case 2.1, (c) Case 2.2, (d) Case 3.1 and (e) Case 3.2. The additional non-optimal DoF tuples are given by $\mathbf{C}_{12}^{[1]} = (1, \lambda_1, 0)$, $\mathbf{C}_{12}^{[2]} = \left(\frac{2-2\lambda_1-\lambda_2}{3}, \frac{2-2\lambda_2-\lambda_1}{3}, 0\right)$, $\mathbf{C}_{12}^{[3]} = (\lambda_2, 1, 0)$, $\mathbf{C}_{12}^{[4]} = \left(1, \frac{1+\lambda_2}{2}, 0\right)$, $\mathbf{C}_{13}^{[1]} = (1, 0, \lambda_1)$, $\mathbf{C}_{13}^{[2]} = \left(\frac{2-2\lambda_1-\lambda_3}{3}, 0, \frac{2-2\lambda_3-\lambda_1}{3}\right)$, $\mathbf{C}_{13}^{[3]} = (\lambda_3, 0, 1)$, $\mathbf{C}_{13}^{[4]} = \left(1, 0, \frac{1+\lambda_3}{2}\right)$ $\mathbf{C}_{23}^{[1]} = (0, 1, \lambda_2)$, $\mathbf{C}_{23}^{[2]} = \left(0, \frac{2-2\lambda_2-\lambda_3}{3}, \frac{2-2\lambda_3-\lambda_2}{3}\right)$, $\mathbf{C}_{23}^{[3]} = (0, \lambda_3, 1)$ and $\mathbf{C}_{23}^{[4]} = \left(0, 1, \frac{1+\lambda_3}{2}\right)$

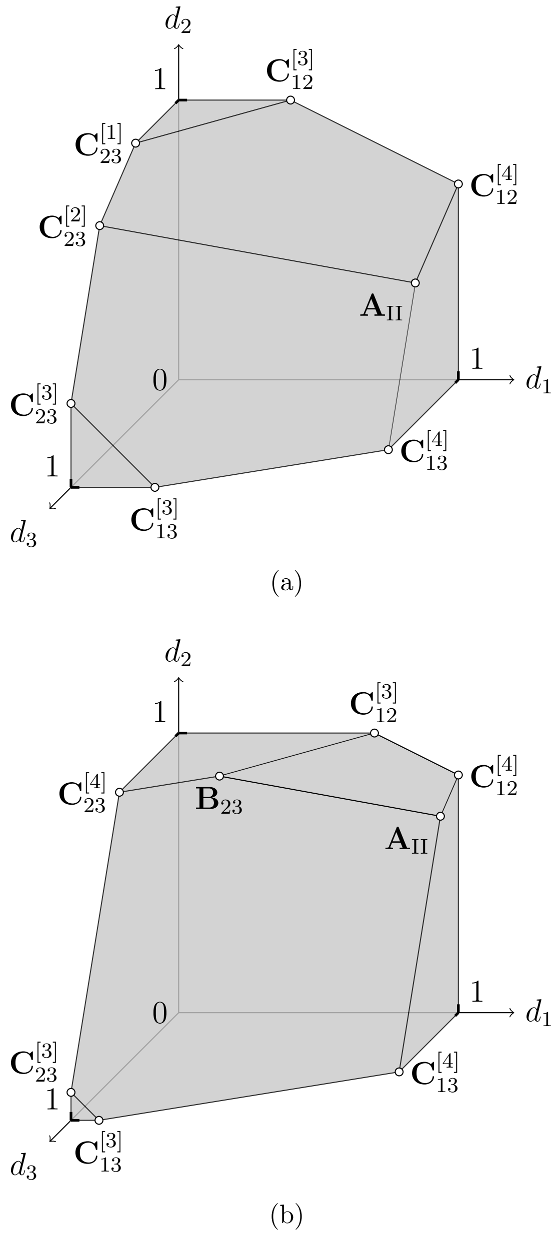

Figure 4.3: Shapes of the DoF regions for Region II: (a) Case 1, (b) Case 2. The additional non-optimal DoF tuples are given by $\mathbf{C}_{12}^{[3]} = (\lambda_2, 1, 0)$, $\mathbf{C}_{12}^{[4]} = \left(1, \frac{1+\lambda_2}{2}, 0\right)$, $\mathbf{C}_{13}^{[3]} = (\lambda_3, 0, 1)$, $\mathbf{C}_{13}^{[4]} = \left(1, 0, \frac{1+\lambda_3}{2}\right)$, $\mathbf{C}_{23}^{[1]} = (0, 1, \lambda_2)$, $\mathbf{C}_{23}^{[2]} = \left(0, \frac{2-2\lambda_2-\lambda_3}{3}, \frac{2-2\lambda_3-\lambda_2}{3}\right)$ and $\mathbf{C}_{23}^{[3]} = (0, \lambda_3, 1)$

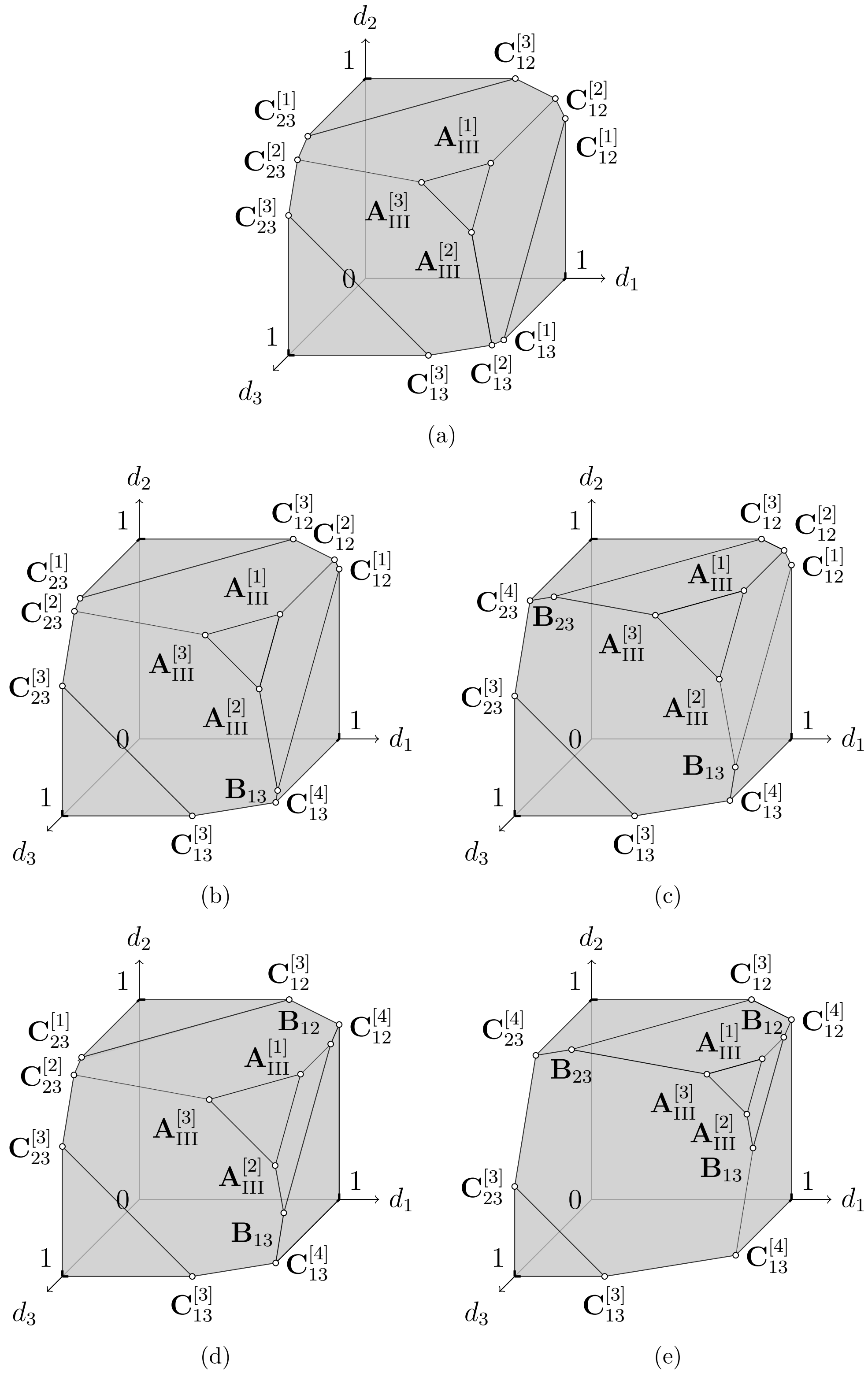

Figure 4.4: Shapes of the DoF regions for Region III: (a) Case 1, (b) Case 2.1, (c) Case 2.2, (d) Case 3.1 and (e) Case 3.2. The additional non-optimal DoF tuples are given by $\mathbf{C}_{12}^{[1]} = (1, \lambda_1, 0)$, $\mathbf{C}_{12}^{[2]} = \left(\frac{2-2\lambda_1-\lambda_2}{3}, \frac{2-2\lambda_2-\lambda_1}{3}, 0\right)$, $\mathbf{C}_{12}^{[3]} = (\lambda_2, 1, 0)$, $\mathbf{C}_{12}^{[4]} = \left(1, \frac{1+\lambda_2}{2}, 0\right)$, $\mathbf{C}_{13}^{[1]} = (1, 0, \lambda_1)$, $\mathbf{C}_{13}^{[2]} = \left(\frac{2-2\lambda_1-\lambda_3}{3}, 0, \frac{2-2\lambda_3-\lambda_1}{3}\right)$, $\mathbf{C}_{13}^{[3]} = (\lambda_3, 0, 1)$, $\mathbf{C}_{13}^{[4]} = \left(1, 0, \frac{1+\lambda_3}{2}\right)$ $\mathbf{C}_{23}^{[1]} = (0, 1, \lambda_2)$, $\mathbf{C}_{23}^{[2]} = \left(0, \frac{2-2\lambda_2-\lambda_3}{3}, \frac{2-2\lambda_3-\lambda_2}{3}\right)$, $\mathbf{C}_{23}^{[3]} = (0, \lambda_3, 1)$ and $\mathbf{C}_{23}^{[4]} = \left(0, 1, \frac{1+\lambda_3}{2}\right)$

4.5.2.2 Achievability

In this section, we show the achievability of the non-optimal DoF region corner points. Below, we provide the CS assignment procedure which achieves the non-optimal DoF corner points for each of the cases given in Section 4.5.2.1.

DoF tuples $\mathbf{B}_{12}$ *and* $\mathbf{B}_{13}$*:* For $\mathbf{B}_{12}$, the encoding over PPD state is performed individually using S_1^2 with the CS fraction $\lambda_{S_1^2} = \lambda_{\text{PPD}}$. For $\mathbf{B}_{13}$, the encoding over PDP state is performed individually using S_2^2 with the CS fraction $\lambda_{S_2^2} = \lambda_{\text{PPD}}$. For the remaining CSIT states, the CS assignment is performed in the following order. Initially, $S_2^{3/2}$ and S_5^2 are applied for joint encoding over PPP, PDD and DDD states. Depending on the relationship between λ_{PPP}, λ_{PDD} and λ_{DDD}, the following three cases are distinguished.

A. $\lambda_{\text{DDD}} - \lambda_{\text{PDD}} \geq 0$

In this case, PDD state can be fully alternated with DDD state using the scheme $S_2^{3/2}$ with the CS fraction $\lambda_{S_2^{3/2}} = 2\lambda_{\text{PDD}}$. The CS assignment for the remaining CSIT states is the following.

$\mathbf{B}_{12}$*:* The remaining DDD state fraction $\lambda^*_{\text{DDD}} = \lambda_{\text{DDD}} - \lambda_{\text{PDD}}$ is alternated with PDP state using the scheme $S_3^{3/2}$ with the CS fraction $\lambda_{S_3^{3/2}} = 2\lambda^*_{\text{DDD}}$. The remaining PDP state fraction is encoded using S_2^2 with the CS fraction $\lambda_{S_2^2} = \lambda_{\text{PDP}} - \lambda^*_{\text{DDD}}$.

$\mathbf{B}_{13}$*:* The remaining DDD state fraction λ^*_{DDD} is alternated with PPD state using the scheme $S_4^{3/2}$ with the CS fraction $\lambda_{S_4^{3/2}} = 2\lambda^*_{\text{DDD}}$. The remaining PPD state fraction is encoded using S_3^2 with the CS fraction $\lambda_{S_3^2} = \lambda_{\text{PPD}} - \lambda^*_{\text{DDD}}$.

B. $\lambda_{\text{PPP}} \geq \lambda_{\text{PDD}} - \lambda_{\text{DDD}} > 0$

In this case, DDD state can be fully alternated with PDD state using the scheme $S_2^{3/2}$ with the CS fraction $\lambda_{S_2^{3/2}} = 2\lambda_{\text{DDD}}$. The remaining PDD state fraction $\lambda^*_{\text{PDD}} = \lambda_{\text{PDD}} - \lambda_{\text{DDD}}$ is alternated with PPP state using the scheme S_5^2 with the CS fraction $\lambda_{S_5^2} = 2\lambda^*_{\text{PDD}}$. The remaining PPP state fraction is encoded using the scheme $\lambda_{S_4^2}$ with the CS fraction $\lambda_{S_5^2} = \lambda_{\text{PPP}} - \lambda^*_{\text{PDD}}$. The CS assignment for the remaining CSIT states is the following.

$\mathbf{B}_{12}$*:* The encoding over PDP state is performed using S_3^2 with the CS fraction $\lambda_{S_3^2} = \lambda_{\text{PDP}}$.

$\mathbf{B}_{13}$*:* The encoding over PPD state is performed using S_2^2 with the CS fraction $\lambda_{S_2^2} = \lambda_{\text{PPD}}$.

C. $\lambda_{\mathrm{PDD}} - \lambda_{\mathrm{DDD}} > \lambda_{\mathrm{PPP}}$

In this case, PPP state can be fully alternated with PDD state using the scheme S_5^2 with the CS fraction $\lambda_{S_5^2} = 2\lambda_{\mathrm{PPP}}$. The remaining PDD state fraction $\lambda_{\mathrm{PDD}}^{**} = \lambda_{\mathrm{PDD}} - \lambda_{\mathrm{PPP}}$ is alternated with DDD state using the schemes $S_1^{5/3}$ and $S_2^{3/2}$ with the CS fractions $\lambda_{S_1^{5/3}} = 3\,(\lambda_{\mathrm{PDD}}^{**} - \lambda_{\mathrm{DDD}})$ and $\lambda_{S_2^{3/2}} = 2\,(2\lambda_{\mathrm{DDD}} - \lambda_{\mathrm{PDD}}^{**})$. The CS assignment for the remaining CSIT states is the following.

$\mathbf{B}_{12}$*:* The encoding over PDP state is performed using S_3^2 with the CS fraction $\lambda_{S_3^2} = \lambda_{\mathrm{PDP}}$.

$\mathbf{B}_{13}$*:* The encoding over PPD state is performed using S_2^2 with the CS fraction $\lambda_{S_2^2} = \lambda_{\mathrm{PPD}}$.

DoF tuple $\mathbf{B}_{23}$*:* The encoding over PPP state is performed individually using the scheme S_4^2 with the CS fraction $\lambda_{S_4^2} = \lambda_{\mathrm{PPP}}$. The encoding over the remaining CSIT states is performed jointly, where PPD state is alternated with PDP, PDD and DDD states using the schemes $S_4^{5/3}$, $S_4^{3/2}$ and $S_5^{3/2}$, where the CS fractions are given by $\lambda_{S_4^{5/3}} = 3\lambda_{\mathrm{PDP}}$, $\lambda_{S_4^{3/2}} = 2\lambda_{\mathrm{DDD}}$ and $\lambda_{S_5^{3/2}} = 2\lambda_{\mathrm{PDD}}$. The remaining PPD state fraction is encoded using the scheme S_1^2 with the CS fraction $\lambda_{S_1^2} = \lambda_{\mathrm{PPD}} - 2\lambda_{\mathrm{PDP}} - \lambda_{\mathrm{PDD}} - \lambda_{\mathrm{DDD}}$.

The summary of the CS assignment for achieving $\mathbf{B}_{12}$, $\mathbf{B}_{13}$ and $\mathbf{B}_{23}$ is given in Tables 4.11 to 4.17.

4.6 Proof of Theorem 4

In this section, the proof of Theorem 4 is given.

$\frac{12}{7}$ DoF for $(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDD}}) = \left(\frac{5}{7}, \frac{1}{7}, \frac{1}{7}\right)$ and $\frac{33}{19}$ DoF for $(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDP}}) = \left(\frac{15}{19}, \frac{2}{19}, \frac{2}{19}\right)$ are achieved using the schemes $S^{12/7}$ and $S^{33/19}$, respectively. For the remaining cases in (4.6), we provide the CS assignment procedure below.

The encoding over PPD, PDP and DPP states is performed independently using the schemes S_1^2, S_2^2 and S_3^2, respectively, with the CS fractions $\lambda_{S_1^2} = \lambda_{\mathrm{PPD}}$, $\lambda_{S_2^2} = \lambda_{\mathrm{PDP}}$ and $\lambda_{S_3^2} = \lambda_{\mathrm{DPP}}$. The encoding over the remaining PPP, PDD, DPD, DDP and DDD states is performed jointly, with the order which we describe next. Initially, the schemes S_5^2, S_6^2 and S_7^2 are applied for joint encoding over the CSIT state pairs (PPP, PDD), (PPP, DPD) and (PPP, DDP), respectively. Depending on whether $\lambda_{\mathrm{PDD}} + \lambda_{\mathrm{DPD}} + \lambda_{\mathrm{DDP}}$ is greater or smaller than λ_{PPP}, the following two cases are distinguished.

Table 4.11: Case A: achieving $\mathbf{B}_{12} = (1, 1-\lambda_1+\lambda_2, 2\lambda_1-\lambda_2-1,)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
$S_{3/2}^2$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$	$2\lambda_{\text{PDD}}$
$S_{3/2}^3$	$(\lambda_{\text{PDP}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$	$2\left(\lambda_{\text{DDD}} - \lambda_{\text{PDD}}\right)$
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	$\lambda_{\text{PDP}} - (\lambda_{\text{DDD}} - \lambda_{\text{PDD}})$

Table 4.12: Case A: achieving $\mathbf{B}_{13} = (1, 2\lambda_1-\lambda_3-1, 1-\lambda_1+\lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
$S_{3/2}^2$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, 0, \frac{1}{2}\right)$	$2\lambda_{\text{PDD}}$
$S_{3/2}^4$	$(\lambda_{\text{PPD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, 0, \frac{1}{2}\right)$	$2\left(\lambda_{\text{DDD}} - \lambda_{\text{PDD}}\right)$
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	$\lambda_{\text{PPD}} - (\lambda_{\text{DDD}} - \lambda_{\text{PDD}})$

Table 4.13: Case B: achieving $\mathbf{B}_{12} = (1, 1-\lambda_1+\lambda_2, 2\lambda_1-\lambda_2-1,)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
$S_{3/2}^2$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$	$2\lambda_{\text{DDD}}$
S_5^2	$(\lambda_{\text{PPP}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\left(\lambda_{\text{PDD}} - \lambda_{\text{DDD}}\right)$
S_4^2	$\lambda_{\text{PPP}} = 1$	$(1,1,0)$	$\lambda_{\text{PPP}} - (\lambda_{\text{PDD}} - \lambda_{\text{DDD}})$

Table 4.14: Case B: achieving $\mathbf{B}_{13} = (1, 2\lambda_1-\lambda_3-1, 1-\lambda_1+\lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
$S_{3/2}^2$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, 0, \frac{1}{2}\right)$	$2\lambda_{\text{DDD}}$
S_5^2	$(\lambda_{\text{PPP}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\left(\lambda_{\text{PDD}} - \lambda_{\text{DDD}}\right)$
S_4^2	$\lambda_{\text{PPP}} = 1$	$(1,0,1)$	$\lambda_{\text{PPP}} - (\lambda_{\text{PDD}} - \lambda_{\text{DDD}})$

Table 4.15: Case C: achieving $\mathbf{B}_{12} = (1, 1-\lambda_1+\lambda_2, 2\lambda_1-\lambda_2-1,)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_5^2	$(\lambda_{\text{PPP}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\text{PPP}}$
$S_4^{5/3}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	$3\left(\lambda_{\text{PDD}} - \lambda_{\text{PPP}} - \lambda_{\text{DDD}}\right)$
$S_4^{3/2}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 0\right)$	$2\left(2\lambda_{\text{DDD}} - (\lambda_{\text{PDD}} - \lambda_{\text{PPP}})\right)$

Table 4.16: Case C: achieving $\mathbf{B}_{13} = (1, 2\lambda_1-\lambda_3-1, 1-\lambda_1+\lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\text{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_5^2	$(\lambda_{\text{PPP}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\text{PPP}}$
$S_1^{5/3}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	$3\left(\lambda_{\text{PDD}} - \lambda_{\text{PPP}} - \lambda_{\text{DDD}}\right)$
$S_2^{3/2}$	$(\lambda_{\text{PDD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, 0, \frac{1}{2}\right)$	$2\left(2\lambda_{\text{DDD}} - (\lambda_{\text{PDD}} - \lambda_{\text{PPP}})\right)$

Table 4.17: Achieving $\mathbf{B}_{23} = (2\lambda_2-\lambda_3-1, 1, 1-\lambda_2+\lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_4^2	$\lambda_{\text{PPP}} = 1$	$(0,1,1)$	λ_{PPP}
$S_{5/3}^4$	$(\lambda_{\text{PPD}}, \lambda_{\text{PDP}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(0, 1, \frac{2}{3}\right)$	$3\lambda_{\text{PDP}}$
$S_{3/2}^4$	$(\lambda_{\text{PPD}}, \lambda_{\text{DDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(0, 1, \frac{1}{2}\right)$	$2\lambda_{\text{DDD}}$
$S_{3/2}^5$	$(\lambda_{\text{PPD}}, \lambda_{\text{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(0, 1, \frac{1}{2}\right)$	$2\lambda_{\text{PDD}}$
S_1^2	$\lambda_{\text{PPD}} = 1$	$(1,1,0)$	$\lambda_{\text{PPD}} - 2\lambda_{\text{PDP}} - \lambda_{\text{PDD}} - \lambda_{\text{DDD}}$

1) Case A: $\lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}} \geq \lambda_{\text{PPP}}$. In this case, PPP state can be fully exhausted using S_5^2, S_6^2 and S_7^2. By denoting the relative fractions of PDD, DPD and DDP states as

$$\gamma_{\text{PDD}} = \frac{\lambda_{\text{PDD}}}{\lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}}}, \tag{4.60}$$

$$\gamma_{\text{DPD}} = \frac{\lambda_{\text{DPD}}}{\lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}}}, \tag{4.61}$$

$$\gamma_{\text{DDP}} = \frac{\lambda_{\text{DDP}}}{\lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}}}, \tag{4.62}$$

$\gamma_{\text{PDD}}+\gamma_{\text{DPD}}+\gamma_{\text{DDP}} = 1$, the CS fractions are given by $\lambda_{S_5^2} = 2\lambda_{\text{PPP}}\gamma_{\text{PDD}}$, $\lambda_{S_6^2} = 2\lambda_{\text{PPP}}\gamma_{\text{DPD}}$ and $\lambda_{S_7^2} = 2\lambda_{\text{PPP}}\gamma_{\text{DDP}}$. The remaining fractions of PDD, DPD and DDP states

$$\lambda^*_{\text{PDD}} = \lambda_{\text{PDD}} - \lambda_{\text{PPP}}\gamma_{\text{PDD}}, \tag{4.63}$$

$$\lambda^*_{\text{DPD}} = \lambda_{\text{DPD}} - \lambda_{\text{PPP}}\gamma_{\text{DPD}}, \tag{4.64}$$

$$\lambda^*_{\text{DDP}} = \lambda_{\text{DDP}} - \lambda_{\text{PPP}}\gamma_{\text{DDP}} \tag{4.65}$$

are alternated with DDD state using the schemes $S_1^{5/3}$, $S_2^{5/3}$ and $S_3^{5/3}$, respectively. Assuming

$$\lambda^*_{\text{PDD}}+\lambda^*_{\text{DPD}}+\lambda^*_{\text{DDP}} = \lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}}-\lambda_{\text{PPP}} \leq 2\lambda_{\text{DDD}} \tag{4.66}$$

holds, the remaining fractions of PDD, DPD and DDP states can be fully exhausted using $S_1^{5/3}$, $S_2^{5/3}$ and $S_3^{5/3}$ with the CS fractions $\lambda_{S_1^{5/3}} = \frac{3}{2}\lambda^*_{\text{PDD}}$, $\lambda_{S_2^{5/3}} = \frac{3}{2}\lambda^*_{\text{DPD}}$ and $\lambda_{S_3^{5/3}} = \frac{3}{2}\lambda^*_{\text{DDP}}$. For encoding over the remaining DDD state fraction, the scheme $S_1^{3/2}$ is applied with the CS fraction $\lambda_{S_1^{3/2}} = \lambda_{\text{DDD}} - \frac{\lambda^*_{\text{PDD}}+\lambda^*_{\text{DPD}}+\lambda^*_{\text{DDP}}}{2}$.

2) Case B: $\lambda_{\text{PDD}}+\lambda_{\text{DPD}}+\lambda_{\text{DDP}} < \lambda_{\text{PPP}}$. In this case, PDD, DPD and DDP states can be fully exhausted using the schemes S_5^2, S_6^2 and S_7^2 with the CS fractions $\lambda_{S_5^2} = 2\lambda_{\text{PDD}}$, $\lambda_{S_6^2} = 2\lambda_{\text{DPD}}$ and $\lambda_{S_7^2} = 2\lambda_{\text{DDP}}$. The remaining PPP state fraction $\lambda^*_{\text{PPP}} = \lambda_{\text{PPP}} - \lambda_{\text{PDD}} - \lambda_{\text{DPD}} - \lambda_{\text{DDP}}$ is alternated with DDD state using the scheme S_8^2. Depending on whether $2\lambda_{\text{DDD}}$ is greater or smaller than λ^*_{PPP}, the following two sub-cases are distinguished.

B.1. $2\lambda_{\text{DDD}} \geq \lambda^*_{\text{PPP}}$: The PPP state fraction can be fully exhausted using S_8^2 with the CS fraction $\lambda_{S_8^2} = \frac{3}{2}\lambda^*_{\text{PPP}}$. Over the remaining fraction of DDD state, encoding using $S_1^{3/2}$ is performed with the CS fraction $\lambda_{S_1^{3/2}} = \lambda_{\text{DDD}} - \frac{\lambda^*_{\text{PPP}}}{2}$.

B.2. $2\lambda_{\text{DDD}} < \lambda^*_{\text{PPP}}$: DDD state can be fully exhausted using $\lambda_{S_8^2}$ with the CS fraction $\lambda_{S_8^2} = 3\lambda_{\text{DDD}}$. Over the remaining PPP state fraction, encoding using S_4^2 is performed with the CS fraction $\lambda_{S_4^2} = \lambda^*_{\text{PPP}} - 2\lambda_{\text{DDD}}$.

The summary of the DoF achievability for cases A, B.1 and B.2 is given in Tables 4.18, 4.19 and 4.20.

Equivalence to (4.5)*:* $2\lambda_{\text{DDD}} < \lambda^*_{\text{PPP}}$ is equivalent to $\lambda_1 + \lambda_2 + \lambda_3 > 2$, hence Case B.2 corresponds to Region III. Consecutively, Cases A.1 and B.1 correspond to Region I for $2\lambda_{\text{DDD}} \geq \lambda_{\text{PDD}} + \lambda_{\text{DPD}} + \lambda_{\text{DDP}} - \lambda_{\text{PPP}} \geq 0$ and $2\lambda_{\text{DDD}} \geq \lambda_{\text{PPP}} - \lambda_{\text{PDD}} - \lambda_{\text{DPD}} - \lambda_{\text{DDP}} > 0$, respectively. Using the relationship between the marginal and joint CSIT state probabilities

$$\lambda_1 = \lambda_{\text{PPP}} + \lambda_{\text{PPD}} + \lambda_{\text{PDP}} + \lambda_{\text{PDD}}, \tag{4.67}$$

$$\lambda_2 = \lambda_{\text{PPP}} + \lambda_{\text{PPD}} + \lambda_{\text{DPP}} + \lambda_{\text{DPD}}, \tag{4.68}$$

$$\lambda_3 = \lambda_{\text{PPP}} + \lambda_{\text{PDP}} + \lambda_{\text{DPP}} + \lambda_{\text{DDP}}, \tag{4.69}$$

yields the required DoF tuples.

4.7 Conclusion

In this chapter, the 2-antenna 3-user MISO BC with alternating CSIT has been considered, where the CSIT for each user can be either P or D. For this network we obtained two new results on the DoF characterization. The first result establishes the DoF region for the case where the admissible CSIT states can take the values PPP, PPD, PDP, PDD and DDD. The second result characterizes the DoF for the case where the CSIT states can take all possible values, however the joint CSIT state probabilities are restricted to fulfil certain relationships. For the achievability, four novel CSs were introduced in which joint encoding over the CSIT state tuples (PPP, PDD), (PDD, DDD), (PDD, DPD, DDD) and (PDD, DPD, DDP) is performed. By assigning the newly proposed and existing CSs to the available CSIT states, optimal DoF were achieved.

Table 4.18: Case A: achieving $\mathbf{A}_{\mathrm{I}} = \left(\frac{1}{2} + \frac{3\lambda_1 - \lambda_2 - \lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_2 - \lambda_1 - \lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_3 - \lambda_1 - \lambda_2}{4}\right)$ for $2\lambda_{\mathrm{DDD}} \geq \lambda_{\mathrm{PDD}} + \lambda_{\mathrm{DPD}} + \lambda_{\mathrm{DDP}} - \lambda_{\mathrm{PPP}} \geq 0$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1, 1, 0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1, 0, 1)$	λ_{PDP}
S_3^2	$\lambda_{\mathrm{DPP}} = 1$	$(0, 1, 1)$	λ_{DPP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PPP}}\gamma_{\mathrm{PDD}}$
S_6^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DPD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PPP}}\gamma_{\mathrm{DPD}}$
S_7^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDP}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 1\right)$	$2\lambda_{\mathrm{PPP}}\gamma_{\mathrm{DDP}}$
$S_1^{5/3}$	$(\lambda_{\mathrm{PDD}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(1, \frac{1}{3}, \frac{1}{3}\right)$	$\frac{3}{2}\lambda^*_{\mathrm{PDD}}$
$S_2^{5/3}$	$(\lambda_{\mathrm{DPD}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, 1, \frac{1}{3}\right)$	$\frac{3}{2}\lambda^*_{\mathrm{DPD}}$
$S_3^{5/3}$	$(\lambda_{\mathrm{DDP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, \frac{1}{3}, 1\right)$	$\frac{3}{2}\lambda^*_{\mathrm{DDP}}$
$S_1^{3/2}$	$\lambda_{\mathrm{DDD}} = 1$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\lambda_{\mathrm{DDD}} - \frac{\lambda^*_{\mathrm{PDD}} + \lambda^*_{\mathrm{DPD}} + \lambda^*_{\mathrm{DDP}}}{2}$

Table 4.19: Case B.1: achieving $\mathbf{A}_{\mathrm{I}} = \left(\frac{1}{2} + \frac{3\lambda_1-\lambda_2-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_2-\lambda_1-\lambda_3}{4}, \frac{1}{2} + \frac{3\lambda_3-\lambda_1-\lambda_2}{4}\right)$ for $2\lambda_{\mathrm{DDD}} \geq \lambda_{\mathrm{PPP}} - \lambda_{\mathrm{PDD}} - \lambda_{\mathrm{DPD}} - \lambda_{\mathrm{DDP}} > 0$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_3^2	$\lambda_{\mathrm{DPP}} = 1$	$(0,1,1)$	λ_{DPP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PDD}}$
S_6^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DPD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1, \frac{1}{2}\right)$	$2\lambda_{\mathrm{DPD}}$
S_7^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDP}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 1\right)$	$2\lambda_{\mathrm{DDP}}$
S_8^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	$\frac{3}{2}\lambda_{\mathrm{PPP}}^*$
$S_1^{3/2}$	$\lambda_{\mathrm{DDD}} = 1$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\lambda_{\mathrm{DDD}} - \frac{\lambda_{\mathrm{PPP}}^*}{2}$

Table 4.20: Case B.2: achieving $\mathbf{A}_{\mathrm{III}}^{[1]} = (\lambda_1, \lambda_2, 2-\lambda_1-\lambda_2)$, $\mathbf{A}_{\mathrm{III}}^{[2]} = (\lambda_1, 2-\lambda_1-\lambda_3, \lambda_3)$ and $\mathbf{A}_{\mathrm{III}}^{[3]} = (2-\lambda_2-\lambda_3, \lambda_2, \lambda_3)$

CS	State fractions	DoF tuples	CS fractions
S_1^2	$\lambda_{\mathrm{PPD}} = 1$	$(1,1,0)$	λ_{PPD}
S_2^2	$\lambda_{\mathrm{PDP}} = 1$	$(1,0,1)$	λ_{PDP}
S_3^2	$\lambda_{\mathrm{DPP}} = 1$	$(0,1,1)$	λ_{DPP}
S_5^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{PDD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1, \frac{1}{2}, \frac{1}{2}\right)$	$2\lambda_{\mathrm{PDD}}$
S_6^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DPD}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1, \frac{1}{2}\right)$	$2\lambda_{\mathrm{DPD}}$
S_7^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDP}}) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 1\right)$	$2\lambda_{\mathrm{DDP}}$
S_8^2	$(\lambda_{\mathrm{PPP}}, \lambda_{\mathrm{DDD}}) = \left(\frac{2}{3}, \frac{1}{3}\right)$	$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$	$3\lambda_{\mathrm{DDD}}$
S_4^2	$\lambda_{\mathrm{PPP}} = 1$	$(1,1,0)$, $(1,0,1)$ and $(0,1,1)$	$\lambda_{\mathrm{PPP}}^* - 2\lambda_{\mathrm{DDD}}$

Chapter 5

Conclusions and Outlook

5.1 Conclusions

In this thesis, DoF of different communication networks with delayed CSIT have been studied. The following three networks have been considered: the MIMO XC with delayed CSIT, the three-user symmetric MIMO IC with delayed CSIT and the 2-antenna 3-user MISO BC with alternating CSIT. For the MIMO XC with delayed CSIT, we performed a decodability analysis of the state-of-the-art transmission scheme. For the three-user symmetric MIMO IC with delayed CSIT, new results on the DoF achievability were proposed. For the 2-antenna 3-user MISO BC with alternating CSIT, partial DoF characterization was provided.

In Chapter 2, the MIMO XC with delayed CSIT is considered. The decodability analysis of the state-of-the-art transmission scheme is performed by studying linear independence of the received linear combinations. For our study, we assume $\min\{M_1, M_2\} > \min\{N_1, N_2\}$ holds. First, we describe the state-of-the-art transmission scheme in detail. Then, the linear independence analysis is performed. To achieve this, an upper bound on the rank of the effective channel matrix is obtained. The upper bound is expressed as two decodability bounds on the parameters of the transmission scheme. In case the proposed bounds override the existing reference bound, linear dependence is stated. To address the issue of linear dependence, a novel transmission scheme is proposed. The parameters of the transmission scheme are chosen to maximize the number of the transmitted information symbols while satisfying the existing and newly proposed decodability bounds. The proposed transmission scheme achieves a number of DoF greater than that of the state-of-the-art transmission scheme in which the number of the transmitted information symbols is reduced to the number of the decodable ones.

In Chapter 3, the three-user symmetric MIMO IC with delayed CSIT is considered. For the considered scenario, two transmission schemes achieving a greater number of DoF are proposed. The transmission schemes have a three-phase structure. For the case $M < N$, the proposed transmission scheme relies on RT-PIN approach. For the design, the redundancy naturally introduced by the channel is taken into account, where the amount of the additionally introduced redundancy in each phase is adjusted according to the ratio $\frac{M}{N}$. For the case $M > N$, the proposed transmission scheme relies on IS-RT-PIN approach. In phase 1, we apply a novel three-part IS-RT. As compared to the existing in the literature work relying on two-part IS-RT, it allows to better overcome the problem of linear dependency. In the proposed IS-RT, IS and RT parts of different transmitters have different durations, which allows to reduce the number of linearly dependent linear combinations. As compared to the existing in the literature scheme where to overcome linear dependence the number of used transmit antennas is limited at all transmitters, in the proposed scheme the number of used transmit antennas is limited at only single transmitter. In phase 2, we rely on IS-RT. The parameters of the transmissions in phases 1 and 2 are chosen to maximize the number of transmitted information symbols while ensuring linear independence of the received linear combinations. Both of the transmission schemes proposed for the cases of $M < N$ and $M > N$ achieve the DoF greater than that in the literature. In addition to the proposed transmission schemes, an upper bound on the linear DoF is proposed, shown to be tight for the regions of antenna configurations of $\frac{1}{2} < \frac{M}{N} \leq \frac{3}{5}$ and $2 \leq \frac{M}{N} < 3$. Our upper bound is based on an upper bound on the ratio of the sizes of signal spaces spanned by two transmitters at intended and unintended receivers.

In Chapter 4, the 2-antenna 3-user MISO BC with alternating CSIT is considered. For the considered scenario, two new results on the DoF characterization are obtained. The first result is the DoF region characterization for the restricted alternating CSIT setting in which the admissible CSIT states are PPP, PPD, PDP, PDD and DDD. The second result is the DoF characterization for the case where the CSIT states can take all possible values, however the joint CSIT state probabilities are restricted to fulfill certain relationship. For the outer bound, we rely on the existing outer bound for delayed and imperfect CSIT. For the achievability, first we propose four novel CSs in which joint encoding over the CSIT state tuples (PPP, PDD), (PDD, DDD), $(\text{PDD}, \text{DPD}, \text{DDD})$ and $(\text{PDD}, \text{DPD}, \text{DDP})$ is performed. After assignment of the newly proposed and existing in the literature CSs to the available CSIT states, optimal DoF are achieved.

5.2 Outlook

Despite the number of results obtained in this thesis, a number of research questions remain unanswered. Firstly, full DoF characterizations for each of the considered networks have not yet been provided. Secondly, the scope of the consideration of this thesis is limited and many other networks could have been considered. In the following, we elaborate more on possible further research directions.

First, for the MIMO XC with delayed CSIT, we proposed the transmission scheme achieving greater number of DoF. However, the DoF achieved by the proposed transmission scheme do not match the existing LDoF upper bound. Closing the gap between the DoF achieved by the transmission scheme and the existing upper bound is a good further topic of investigation. Moreover, the existing upper bound restricts the transmitters to use only linear encoding strategies. Relaxing this assumption and obtaining an upper bound in terms of DoF, instead of LDoF, would be another problem to consider.

Next, we considered the 3-user symmetric MIMO IC with delayed CSIT. This setting is challenging as even in the SISO case, optimal DoF are not known yet. In this thesis, we managed only to obtain partial LDoF characterization for the case $\frac{M}{N} < 1$. Novel upper bounds, as well as novel transmission schemes, are worth studying in order to extend the region where optimal DoF are known beyond that obtained in this thesis. It needs to be noted that RT-PIN and IS-RT-PIN techniques used in this thesis can be potentially adapted to other networks where interference at receivers is due to multiple transmitters. In particular, we believe that designing new DoF achievability schemes for the 3-user symmetric MIMO XC would be a good further topic of investigation. Another possible direction is to consider an arbitrary number K of users, which could be done for both IC and XC models.

Last, we considered the 2-antenna 3-user MISO BC with alternating CSIT. For this scenario, full DoF region characterization has been obtained for the case where the CSIT states are restricted. For the non-restricted case, only partial DoF characterization has been provided. Full DoF characterization for the non-restricted case remains yet an open problem. Increasing the number of transmit antennas, as well as number of users, could be another topic of investigation. To date, for the overloaded MISO BC with simply delayed CSIT, optimal DoF are not known other than that for the case $M = 2$, $K = 3$. Finding the optimal DoF of the overloaded MISO BC with delayed CSIT with arbitrary numbers of antennas and users is then the first problem to consider before proceeding with more sophisticated alternating CSIT settings. Another possible direction is to consider the more general delayed and imperfect CSIT setting where in addition to the delayed CSIT, the transmitter has instantaneous CSIT knowledge with limited precision. Providing optimal DoF for the $M = 2$, $K = 3$ MISO BC with delayed and imperfect CSIT would be then the first problem to consider.

Chapter A

Appendix

A.1 Proof of Theorem 2

In this section, we prove Theorem 2 given in Section 3.3:

Theorem 2. *For the 3-user MIMO IC with delayed CSIT, the linear DoF are upper bounded as*

$$d_{\text{lin}} \leq \begin{cases} \frac{3MN}{M+N}, & \frac{1}{2} < \frac{M}{N} \leq \frac{3}{4}, \\ \frac{9N}{7}, & \frac{3}{4} < \frac{M}{N} \leq 1, \\ \frac{9MN}{5M+2N}, & 1 < \frac{M}{N} \leq 2, \\ \frac{3N}{2}, & 2 < \frac{M}{N} < 3. \end{cases} \tag{A.1}$$

The proof is inspired by the technique used to prove the upper bound on the linear sum-DoF of the three-user SISO IC in [LAS14], which relies on the upper bound on the ratio Γ between the sizes of the signal spaces spanned by the signals of two transmitters at useful and unintended receivers. Our proof extends the proof given in [LAS14] to a MIMO setting, where for the upper bound on Γ, we refer to the results obtained in [KA14].

Proof. Since the channel is symmetric, the optimum DoF tuple satisfies $d_1 = d_2 = d_3$, with $b_1(n) = b_2(n) = b_3(n) = b(n)$. To prove (A.1), it suffices to obtain a bound

$$\frac{b(n)}{n} \leq \begin{cases} \frac{MN}{M+N}, & \frac{1}{2} < \frac{M}{N} \leq \frac{3}{4}, \\ \frac{3N}{7}, & \frac{3}{4} < \frac{M}{N} \leq 1, \\ \frac{3MN}{5M+2N}, & 1 < \frac{M}{N} \leq 2, \\ \frac{N}{2}, & 2 < \frac{M}{N} < 3. \end{cases} \tag{A.2}$$

We start with the equality which holds given $\mathbf{u}_1$ is decodable at Rx_1:

$$
\begin{aligned}
b(n) &= \mathrm{rank}\left(\mathbf{H}_{11}^n \mathbf{C}_1^n\right) \\
&\overset{\text{a.s.}}{=} \mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{11}^n \mathbf{C}_1^n & \mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right) - \mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right) \quad \text{(A.3)} \\
&\leq nN - \mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right), \quad \text{(A.4)}
\end{aligned}
$$

where (A.3) follows from Lemma 3 in [LAS14] and (A.4) is due to $\mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{11}^n \mathbf{C}_1^n & \mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right) \leq nN$.

$\mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right)$ denoting the size of the signal sub-space spanned by the interference signals at Rx_1 can be lower-bounded using (13) in [KA14] as

$$
\begin{aligned}
\mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{12}^n \mathbf{C}_2^n & \mathbf{H}_{13}^n \mathbf{C}_3^n\end{bmatrix}\right) &\geq \frac{1}{\Gamma_{\max}} \mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{22}^n \mathbf{C}_2^n & \mathbf{H}_{33}^n \mathbf{C}_3^n\end{bmatrix}\right) \\
&\overset{\text{a.s.}}{=} \frac{1}{\Gamma_{\max}} \left(\mathrm{rank}\left(\mathbf{H}_{22}^n \mathbf{C}_2^n\right) + \mathrm{rank}\left(\mathbf{H}_{33}^n \mathbf{C}_3^n\right)\right) \quad \text{(A.5)} \\
&= \frac{2b(n)}{\Gamma_{\max}}, \quad \text{(A.6)}
\end{aligned}
$$

where (A.5) is due to independence of $\mathbf{C}_2^n$ and $\mathbf{C}_3^n$ on $\mathbf{H}_{22}^n$ and $\mathbf{H}_{33}^n$ and (A.6) follows from decodability of $\mathbf{u}_2$ and $\mathbf{u}_3$ at Rx_2 and Rx_3, respectively. For the upper bound $\Gamma_{\max}$, we refer to the results in [KA14] of

$$
\Gamma_{\max} = \begin{cases} \frac{2M}{N}, & \frac{1}{2} < \frac{M}{N} \leq \frac{3}{4}, \\ \frac{3}{2}, & \frac{3}{4} < \frac{M}{N} \leq 1, \\ \frac{3M}{M+N}, & 1 < \frac{M}{N} \leq 2, \\ 2, & 2 < \frac{M}{N} < 3. \end{cases} \quad \text{(A.7)}
$$

Substituting (A.6) into (A.4) yields

$$
\frac{b(n)}{n} \leq \frac{N}{1 + \frac{2}{\Gamma_{\max}}}, \quad \text{(A.8)}
$$

where from inserting (A.7) into (A.8), (A.2) follows, completing thus the proof.

□

A.2 Proof of Lemma 2

In this section, we prove Lemma 2 given in Section 3.4.3.2:

Lemma 2. $\mathbf{u}_1^{(1)}$ *and* $\mathbf{u}_2^{(1)}$ *are decodable only if the following bound holds:*

$$\frac{9N-2M}{4N}\frac{T^{(1,1)}}{T^{(1)}}+\frac{T^{(1,2)}}{T^{(1)}}\geq 1. \tag{A.9}$$

The proof obtains an upper bound on the rank of a matrix, the rows of which are comprised of the coefficients of the linear combinations of $\mathbf{u}_1^{(1,1)}$ obtained by Rx_1 from order-2 symbol vectors $\mathbf{u}_{3|1,3}$, $\mathbf{u}_{2|1,2}$, $\mathbf{u}_{1|1,2}$ and $\mathbf{u}_{1|1,3}$. A similar statement holds for Rx_2 due to symmetry.

First, we construct the matrix of the linear combinations of $\mathbf{u}_1^{(1,1)}$ obtained from order-2 symbols.

Linear combinations of $\mathbf{u}_1^{(1,1)}$ *obtained from* $\mathbf{u}_{3|1,3}$*:* $\mathbf{u}_{3|1,3}=\mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{13}^{(1)}\mathbf{u}_3^{(1)}$ can be used to cancel the residual interference in

$$\mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{y}_1^{(1)}=\mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)}+\mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{13}^{(1)}\mathbf{u}_3^{(1)} \tag{A.10}$$

to obtain $\mathbf{W}_{12}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)}$.

By expressing the precoding matrix used by Tx_1 as

$$\mathbf{V}_{2,3;1}^{(1,l)}=\mathbf{V}_{21}^{(1,l)\mathrm{H}}\mathbf{H}_{21}^{(1,l)}, \tag{A.11}$$

$l\in\{1,2\}$, we write $\bar{\mathbf{H}}_{11}^{(1)}$ as

$$\bar{\mathbf{H}}_{11}^{(1)}=\begin{bmatrix}\mathbf{H}_{11}^{(1,1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{11}^{(1,2)}\\ \mathbf{H}_{11}^{(1,3)}\mathbf{C}_1^{(1,3,1)}\mathbf{V}_{21}^{(1,1)\mathrm{H}}\mathbf{H}_{21}^{(1,1)} & \mathbf{H}_{11}^{(1,3)}\mathbf{C}_1^{(1,3,2)}\mathbf{V}_{21}^{(1,2)\mathrm{H}}\mathbf{H}_{21}^{(1,2)}\end{bmatrix}. \tag{A.12}$$

To ensure orthogonality to $\bar{\mathbf{H}}_{12}^{(1)}$ which was given in (3.51), without loss of generality, $\mathbf{W}_{12}^{\mathrm{H}}$ can be assumed to have the form of

$$\mathbf{W}_{12}^{(1)\mathrm{H}}=\begin{bmatrix}-\mathbf{H}_{12}^{(1,3)}\mathbf{C}_2^{(1,3,1)}\mathbf{V}_{12}^{(1,1)\mathrm{H}} & -\mathbf{H}_{12}^{(1,3)}\mathbf{C}_2^{(1,3,2)}\mathbf{V}_{12}^{(1,2)\mathrm{H}} & \mathbf{I}_{NT^{(1,3)}}\end{bmatrix}. \tag{A.13}$$

Let us denote the matrices comprised of the first $MT^{(1,1)}$ and last $MT^{(1,2)}$ rows of $\bar{\mathbf{H}}_{11}^{(1)}$ as $\bar{\mathbf{H}}_{11}^{(1)[1]}\in\mathbb{C}^{NT^{(1)}\times MT^{(1,1)}}$ and $\bar{\mathbf{H}}_{11}^{(1)[2]}\in\mathbb{C}^{NT^{(1)}\times MT^{(1,2)}}$, respectively, where

$$\bar{\mathbf{H}}_{11}^{(1)}=\begin{bmatrix}\bar{\mathbf{H}}_{11}^{(1)[1]} & \bar{\mathbf{H}}_{11}^{(1)[2]}\end{bmatrix} \tag{A.14}$$

and

$$\mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{H}_{11}^{(1)}\mathbf{u}_1 = \mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{H}_{11}^{(1)[1]}\mathbf{u}_1^{(1,1)} + \mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{H}_{11}^{(1)[2]}\mathbf{u}_1^{(1,2)}, \tag{A.15}$$

From (A.12) and (A.13), the matrix of the obtained linear combinations of $\mathbf{u}_1^{(1,1)}$ is evaluated as

$$\mathbf{W}_{12}^{(1)\mathrm{H}}\mathbf{H}_{11}^{(1)[1]} = -\mathbf{H}_{12}^{(1,3)}\mathbf{C}_2^{(1,3,1)}\mathbf{V}_{12}^{(1,1)}\mathbf{H}_{11}^{(1,1)} + \mathbf{H}_{11}^{(1,3)}\mathbf{C}_1^{(1,3,2)}\mathbf{V}_{21}^{(1,2)}\mathbf{H}_{21}^{(1,1)}. \tag{A.16}$$

Linear combinations of $\mathbf{u}_1^{(1,1)}$ *obtained from* $\mathbf{u}_{2|1,2}$*:* $\mathbf{u}_{2|1,2} = \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{12}^{(1)}\mathbf{u}_3^{(1)}$ can be used to cancel the residual interference in

$$\mathbf{W}_{13}^{(1)\mathrm{H}}\mathbf{y}_1^{(1)} = \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)} + \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{12}^{(1)}\mathbf{u}_2^{(1)} \tag{A.17}$$

to obtain

$$\mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)}\mathbf{u}_1^{(1)} = \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)[1]}\mathbf{u}_1^{(1,1)} + \mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)[2]}\mathbf{u}_1^{(1,2)}. \tag{A.18}$$

To ensure orthogonality to $\bar{\mathbf{H}}_{13}^{(1)}$ which was given in (3.54), without loss of generality, $\mathbf{W}_{13}^{(1)\mathrm{H}}$ can be assumed to have the form

$$\mathbf{W}_{13}^{(1)\mathrm{H}} = \begin{bmatrix} -\mathbf{H}_{13}^{(1,2)}\mathbf{C}_3^{(1,2)}\mathbf{V}_{13}^{(1)} & \mathbf{I}_{NT^{(1,2)}} & \mathbf{0} \\ -\mathbf{H}_{13}^{(1,3)}\mathbf{C}_3^{(1,3)}\mathbf{V}_{13}^{(1)} & \mathbf{0} & \mathbf{I}_{NT^{(1,3)}} \end{bmatrix}. \tag{A.19}$$

From (A.12) and (A.19), the matrix of the obtained linear combinations of $\mathbf{u}_1^{(1,1)}$ is evaluated as

$$\mathbf{W}_{13}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{11}^{(1)[1]} = \begin{bmatrix} -\mathbf{H}_{13}^{(1,2)}\mathbf{C}_3^{(1,2)}\mathbf{V}_{13}^{(1)}\mathbf{H}_{11}^{(1,1)} \\ -\mathbf{H}_{13}^{(1,3)}\mathbf{C}_3^{(1,3)}\mathbf{V}_{13}^{(1)}\mathbf{H}_{11}^{(1,1)} + \mathbf{H}_{11}^{(1,3)}\mathbf{C}_1^{(1,3,1)}\mathbf{V}_{21}^{(1,1)}\mathbf{H}_{21}^{(1,1)} \end{bmatrix}. \tag{A.20}$$

Linear combinations of $\mathbf{u}_1^{(1,1)}$ *obtained from* $\mathbf{u}_{1|1,2}$ *and* $\mathbf{u}_{1|1,3}$*:*

$$\begin{aligned} \mathbf{u}_{1|1,2}^{(1)} &= \mathbf{W}_{23}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{21}^{(1)}\mathbf{u}_1^{(1)} = \mathbf{W}_{23}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{21}^{(1)[1]}\mathbf{u}_1^{(1,1)} + \mathbf{W}_{23}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{21}^{(1)[2]}\mathbf{u}_1^{(1,2)}, \\ \mathbf{u}_{1|1,3}^{(1)} &= \mathbf{W}_{32}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{31}^{(1)}\mathbf{u}_1^{(1)} = \mathbf{W}_{32}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{31}^{(1)[1]}\mathbf{u}_1^{(1,1)} + \mathbf{W}_{32}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{31}^{(1)[2]}\mathbf{u}_1^{(1,2)} \end{aligned} \tag{A.21}$$

are directly linear combinations $\mathbf{u}_1^{(1,1)}$ and $\mathbf{u}_1^{(1,2)}$, where $\bar{\mathbf{H}}_{21}^{(1)[1]}, \bar{\mathbf{H}}_{31}^{(1)[1]} \in \mathbb{C}^{NT^{(1)}\times MT^{(1,1)}}$ are the matrices comprised of the first $MT^{(1,1)}$ columns of $\bar{\mathbf{H}}_{21}^{(1)}, \bar{\mathbf{H}}_{31}^{(1)}$ and $\bar{\mathbf{H}}_{21}^{(1)[2]}, \bar{\mathbf{H}}_{31}^{(1)[2]} \in \mathbb{C}^{NT^{(1)}\times MT^{(1,2)}}$ are the matrices comprised of the last $MT^{(1,2)}$ columns of $\bar{\mathbf{H}}_{21}^{(1)}, \bar{\mathbf{H}}_{31}^{(1)}$, respectively.

$\mathbf{W}_{23}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{21}^{(1)[1]}$ and $\mathbf{W}_{32}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{31}^{(1)[1]}$ can be evaluated similarly to (A.16) and (A.20) as

$$\mathbf{W}_{23}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{21}^{(1,1)[1]} = \begin{bmatrix} -\mathbf{H}_{23}^{(1,2)}\mathbf{C}_3^{(1,2)}\mathbf{V}_{23}^{(1)}\mathbf{H}_{21}^{(1,1)} \\ \mathbf{H}_{21}^{(1,3)}\mathbf{C}_1^{(1,3,1)}\mathbf{V}_{21}^{(1,1)}\mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{23}^{(1,3)}\mathbf{C}_3^{(1,3)}\mathbf{V}_{23}^{(1)}\mathbf{H}_{21}^{(1,1)} \end{bmatrix}, \tag{A.22}$$

$$\mathbf{W}_{32}^{(1)\mathrm{H}}\bar{\mathbf{H}}_{31}^{(1,1)[1]} = \begin{bmatrix} \mathbf{H}_{31}^{(1,3)}\mathbf{C}_1^{(1,3,1)}\mathbf{V}_{21}^{(1,1)}\mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{32}^{(1,3)}\mathbf{C}_2^{(1,3,1)}\mathbf{V}_{32}^{(1,1)}\mathbf{H}_{31}^{(1,1)} \end{bmatrix}, \tag{A.23}$$

where for brevity, the details are omitted.

Linear independence analysis: We collect the matrices containing the coefficients of $\mathbf{u}_1^{(1,1)}$ into a matrix

$$\bar{\mathbf{H}}_1^{(1)} = \begin{bmatrix} \mathbf{W}_{12}^{(1)\mathrm{H}} \bar{\mathbf{H}}_{11}^{(1)[1]} \\ \mathbf{W}_{13}^{(1)\mathrm{H}} \bar{\mathbf{H}}_{11}^{(1)[1]} \\ \mathbf{W}_{23}^{(1)\mathrm{H}} \bar{\mathbf{H}}_{21}^{(1)[1]} \\ \mathbf{W}_{32}^{(1)\mathrm{H}} \bar{\mathbf{H}}_{31}^{(1)[1]} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11}^{(1,3)} \mathbf{C}_1^{(1,3,1)} \mathbf{V}_{21}^{(1,1)} \mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{12}^{(1,3)} \mathbf{C}_2^{(1,3,1)} \mathbf{V}_{12}^{(1,1)} \mathbf{H}_{11}^{(1,1)} \\ -\mathbf{H}_{13}^{(1,2)} \mathbf{C}_3^{(1,3,2)} \mathbf{V}_{13}^{(1)} \mathbf{H}_{11}^{(1,1)} \\ \mathbf{H}_{11}^{(1,3)} \mathbf{C}_1^{(1,3,1)} \mathbf{V}_{21}^{(1,1)} \mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{13}^{(1,3)} \mathbf{C}_3^{(1,3)} \mathbf{V}_{13}^{(1)} \mathbf{H}_{11}^{(1,1)} \\ -\mathbf{H}_{23}^{(1,2)} \mathbf{C}_3^{(1,2)} \mathbf{V}_{23}^{(1,1)} \mathbf{H}_{21}^{(1,1)} \\ \mathbf{H}_{21}^{(1,3)} \mathbf{C}_1^{(1,3,1)} \mathbf{V}_{21}^{(1,1)} \mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{23}^{(1,3)} \mathbf{C}_3^{(1,3)} \mathbf{V}_{23}^{(1)} \mathbf{H}_{21}^{(1,1)} \\ \mathbf{H}_{31}^{(1,3)} \mathbf{C}_1^{(1,3,1)} \mathbf{V}_{21}^{(1,1)} \mathbf{H}_{21}^{(1,1)} - \mathbf{H}_{32}^{(1,3)} \mathbf{C}_2^{(1,3,1)} \mathbf{V}_{32}^{(1,1)} \mathbf{H}_{31}^{(1,1)} \end{bmatrix}. \tag{A.24}$$

In the following, we obtain an upper bound on the rank of $\bar{\mathbf{H}}_1^{(1)}$, which is written by combining the matrices having common terms as

$$\begin{aligned} \bar{\mathbf{H}}_1^{(1)} = & \begin{bmatrix} -\mathbf{H}_{12}^{(1,3)} \mathbf{C}_2^{(1,3,1)} \mathbf{V}_{12}^{(1,1)} \\ \begin{bmatrix} -\mathbf{H}_{13}^{(1,2)} \mathbf{C}_3^{(1,2)} \\ -\mathbf{H}_{13}^{(1,3)} \mathbf{C}_3^{(1,3)} \end{bmatrix} \mathbf{V}_{13}^{(1)} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{H}_{11}^{(1,1)} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{H}_{32}^{(1,3)} \mathbf{C}_2^{(1,3,1)} \mathbf{V}_{32}^{(1,1)} \end{bmatrix} \mathbf{H}_{31}^{(1,1)} \\ & + \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{H}_{23}^{(1,2)} \mathbf{C}_3^{(1,2)} \\ -\mathbf{H}_{23}^{(1,2)} \mathbf{C}_3^{(1,3)} \\ \mathbf{0} \end{bmatrix} \mathbf{V}_{23}^{(1)} + \begin{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{H}_{11}^{(1,3)} \\ \mathbf{0} \\ \mathbf{H}_{21}^{(1,3)} \\ \mathbf{H}_{31}^{(1,3)} \end{bmatrix} \mathbf{C}_1^{(1,3,1)} \mathbf{V}_{21}^{(1,1)} \right) \mathbf{H}_{21}^{(1,1)}. \end{aligned} \tag{A.25}$$

Using rank properties of sums and products of matrices we have

$$\begin{aligned} & \operatorname{rank}\left(\bar{\mathbf{H}}_1^{(1,1)}\right) \\ & \leq \min(NT^{(1,1)}, \\ & \quad \min\{(2N-M)T^{(1,1)}, NT^{(1,3)}\} + \min\{2N(T^{(1,1)} - 2T^{(1,3)}), N(T^{(1,2)} + T^{(1,3)})\}) \\ & + \min\{NT^{(1,1)}, (2N-M)T^{(1,1)}, NT^{(1,3)}\} \\ & + \min(NT^{(1,1)}, \\ & \quad \min\{(2N-M)T^{(1,1)}, MT^{(1,3)}, 3NT^{(1,3)}\} \\ & \quad + \min\{2N(T^{(1,1)} - 2T^{(1,3)}), N(T^{(1,2)} + T^{(1,3)}\}) \\ & \leq 3(2N-M)T^{(1,1)} + 4N(T^{(1,1)} - 2T^{(1,3)}). \end{aligned} \tag{A.26}$$

For the decodability, the upper bound in (A.26) has to be greater than or equal to the maximum rank $MT^{(1)}$ of $\bar{\mathbf{H}}_1^{(1)}$, which yields the decodability bound in (A.9).

A.3 Proof of Lemma 3

In this section, we prove Lemma 1 given in Section 2.4:

Lemma 3. $\mathbf{u}_{1|1,2}^{(2)}$ *and* $\mathbf{u}_{2|1,2}^{(2)}$ *are decodable only if the following bound holds:*

$$\frac{T^{(2,1)}}{T^{(2)}} \geq \frac{2N-M}{4N-M}. \tag{A.27}$$

The proof follows the footsteps of the proof of Lemma 1 given in Section 2.4. The proof obtains an upper bound on the rank of the matrix whose rows contain the coefficients of the linear combinations of $\mathbf{u}_{1|1,2}^{(2)}$ and $\mathbf{u}_{2|1,2}^{(2)}$ which will be used for decoding at Rx_1. A similar statement holds for Rx_2 due to symmetry.

Constructing matrix of linear combinations: From $\mathbf{y}_1^{(2)}$ and $\mathbf{u}_{1|1,2;3} = \mathbf{W}_3^{(2)\mathrm{H}}\mathbf{H}_{31}^{(2)}\mathbf{u}_{1|1,2}^{(2)}$, we construct a vector containing linear combinations used by Rx_1 for decoding:

$$\begin{bmatrix} \mathbf{y}_1^{(2)} \\ \mathbf{u}_{1|1,2;3} \end{bmatrix} = \bar{\mathbf{H}}_1^{(2)} \begin{bmatrix} \mathbf{u}_{1,2|1}^{(2,1)} \\ \mathbf{u}_{1,2|1}^{(2,2)} \\ \mathbf{u}_{1,2|2}^{(2)} \end{bmatrix} \in \mathbb{C}^{N\left(T^{(2)}+T^{(2,2)}\right)\times 1}, \tag{A.28}$$

where $\bar{\mathbf{H}}_1 \in \mathbb{C}^{N\left(T^{(2)}+T^{(2,2)}\right)\times MT^{(2)}+b_2^{(2)}}$ is the effective channel matrix.

To ensure orthogonality to $\mathbf{H}_{31}^{(2)}$ which was given in (3.72), without loss of generality, $\mathbf{W}_3^{(2)\mathrm{H}}$ is assumed to have the form

$$\mathbf{W}_3^{(2)\mathrm{H}} = \begin{bmatrix} -\mathbf{H}_{32}^{(2,2)}\mathbf{C}_{2|1,2}^{(2,2)} & \mathbf{I}_{NT^{(2)}} \end{bmatrix}. \tag{A.29}$$

The matrix of the linear combinations is then evaluated as

$$\bar{\mathbf{H}}_1^{(2)} = \begin{bmatrix} \mathbf{H}_{11}^{(2,1)} & \mathbf{0} & \mathbf{H}_{12}^{(2,1)}\mathbf{C}_{2|1,2}^{(2,1)} \\ \mathbf{0} & \mathbf{H}_{11}^{(2,2)} & \mathbf{H}_{12}^{(2,2)}\mathbf{C}_{2|1,2}^{(2,2)}\mathbf{H}_{22}^{(2,1)}\mathbf{C}_2^{(2,1)} \\ -\mathbf{H}_{32}^{(2,2)}\mathbf{C}_{2|1,2}^{(2,2)}\mathbf{H}_{31}^{(2,1)} & \mathbf{H}_{31}^{(2,2)} & \mathbf{0} \end{bmatrix}. \tag{A.30}$$

Linear independence analysis: In the following we obtain an upper bound on the rank of matrix $\bar{\mathbf{H}}_1^{(2)\prime} \in \mathbb{C}^{2NT^{(2,2)} \times MT^{(2)} + b_2^{(2)}}$ comprised of the last $2NT^{(2,2)}$ rows of the matrix $\bar{\mathbf{H}}_1^{(2)}$ given by

$$\bar{\mathbf{H}}_1^{(2)\prime} = \begin{bmatrix} \mathbf{0} & \mathbf{H}_{11}^{(2,2)} & \mathbf{H}_{12}^{(2,2)} \mathbf{C}_{2|1,2}^{(2,2)} \mathbf{H}_{32}^{(2,1)} \mathbf{C}_{2|1,2}^{(2,1)} \\ -\mathbf{H}_{32}^{(2,2)} \mathbf{C}_{2|1,2}^{(2,2)} \mathbf{H}_{31}^{(2,1)} & \mathbf{H}_{31}^{(2,2)} & \mathbf{0} \end{bmatrix}. \tag{A.31}$$

For the rank of $\bar{\mathbf{H}}_1^{(2)\prime}$ we have

$$\begin{aligned} \operatorname{rank}\left(\bar{\mathbf{H}}_1^{(2)\prime}\right) \quad & \leq \operatorname{rank}\left(\mathbf{H}_{32}^{(2,2)} \mathbf{C}_{2|1,2}^{(2,2)} \mathbf{H}_{31}^{(2,1)}\right) \\ & + \operatorname{rank}\left(\begin{bmatrix} \mathbf{H}_{11}^{(2,2)} \\ \mathbf{H}_{31}^{(2,2)} \end{bmatrix}\right) + \operatorname{rank}\left(\mathbf{H}_{12}^{(2)} \mathbf{C}_{2|1,2}^{(2,2)} \mathbf{H}_{32}^{(2,1)} \mathbf{C}_{2|1,2}^{(2,1)}\right). \\ & \leq \min\left(NT^{(2,1)}, NT^{(2,2)}\right) + \\ & + MT^{(2,2)} + \min\left(NT^{(2,2)}, MT^{(2,2)}, NT^{(2,1)}\right) \\ & \leq N \min\left(T^{(2)}, 2T^{(2,2)}\right) + MT^{(2,2)}. \end{aligned} \tag{A.32}$$

For the decodability, the upper bound of (A.32) has to be greater than or equal to the maximum rank $2NT^{(2,2)}$ of $\bar{\mathbf{H}}_1^{(2)\prime}$, which yields the decodability bound in (A.27).

List of Acronyms

SISO Single-Input Single-Output

MISO Multiple-Input Single-Output

MIMO Multiple-Input Multiple-Output

BC Broadcast Channel

XC X-Channel

CSI Channel State Information

CSIT Channel State Information at the Transmitter

SNR Signal-to-Noise Ratio

DoF Degrees of Freedom

LDoF Linear Degrees of Freedom

ZF Zero-Forcing

MAT Maddah-Ali and Tse Scheme

RT Redundancy Transmission

PIN Partial Interference Nulling

IS Interference Sensing

CS Constituent Encoding Scheme

i.i.d. independently and identically distributed

a.s. almost surely

List of Symbols

P	Transmit power
M	Number of transmit antennas
N	Number of receive antennas
K	Number of receivers
$\mathbf{x}_i(t)$	Signal vector transmitted by Tx_i
$\mathbf{y}_j(t)$	Signal vector received by Rx_j
$\mathbf{H}_{ji}(t)$	Channel matrix between Tx_i and Rx_j
$\mathbf{z}_j(t)$	Receive noise vector at Rx_j
$\mathcal{H}^t$	Set of channel matrices up to the t-th channel use
n	Communication duration
$\mathbf{x}_i^n$	Vertical concatenation of signals transmitted by Tx_i over communication duration n
$\mathbf{y}_j^n$	Vertical concatenation of signals received by Rx_j over communication duration n
$\mathbf{H}_{ji}^n$	Diagonal concatenation of the channel matrices between Tx_i and Rx_j for communication duration n
$\mathbf{z}_j^n$	Vertical concatenation of the noise vectors received by Rx_j over communication duration n
$\mathbf{I}_k$	Identity matrix of size k
$\mathcal{CN}(\mathbf{0}, \mathbf{I}_k)$	k-dimensional complex Gaussian distribution with zero-mean and identity covariance matrix
d_i	Achievable DoF for Rx_i
d_Σ	Achievable sum-DoF
$\mathcal{D}$	DoF region
$\mathcal{D}_{\mathrm{lin}}$	Linear DoF region
d_{lin}	Maximum linearly achievable DoF
d	Maximum achievable DoF
$T_\Sigma^{(l)}$	Total duration of phase l
$k^{(l)}$	Number of transmission blocks of phase l
$T^{(l)}$	Duration of transmission block of phase l
$T^{(l,k)}$	Duration of transmission block of part k of phase l
$\mathbf{x}_i^{(l)}$	Signal transmitted by Tx_i in phase l
$\mathbf{x}_i^{(l,k)}$	Signal transmitted by Tx_i in part k of phase l
$\mathbf{y}_j^{(l)}$	Signal received by Rx_j in phase l

$\mathbf{y}_j^{(l,k)}$	Signal received by Rx_j in part k of phase l
$\mathbf{H}_{ji}^{(l)}$	Channel matrix between Tx_i and Rx_j in phase l
$\mathbf{H}_{ji}^{(l,k)}$	Channel matrix between Tx_i and Rx_j in part k of phase l
$\bar{\mathbf{H}}_{ji}^{(l)}$	Effective channel matrix between Tx_i and Rx_j in phase l
$\mathbf{W}_{ji}^{(l)}$	Projection matrix using which Rx_j cancels the signal of Tx_i in phase l

The MIMO X-Channel:

M_i	Number of transmit antennas at Tx_i
N_j	Number of receive antennas at Rx_j
$b_{ji}(n)$	Number of information symbols transmitted from Tx_i to Rx_j over communication duration n
$\mathbf{u}_{ji}$	Symbol vector transmitted from Tx_i to Rx_j
$\mathbf{C}_{ji}(t)$	Precoding matrix used by Tx_i for transmission to Rx_j in the t-th channel use
$\mathbf{C}_{ji}^n$	Vertical concatenation of precoding matrices used by Tx_i for transmission to Rx_j over communication duration n
$\mathcal{I}_{ji}$	Subspace at Rx_j spanned by the signals interfering with $\mathbf{u}_{ji}$
d_{ji}	Achievable DoF for the communication from Tx_i to Rx_j
$b_{ji}^{(l)}$	Number of information transmitted from Tx_i to Rx_j in phase l
$\mathbf{u}_{ji}^{(l)}$	Symbol vector transmitted from Tx_i to Rx_j in phase l
$\mathbf{u}_{ji}^{(l,k)}$	Symbol vector transmitted from Tx_i to Rx_j in part k of phase l
$\mathbf{C}_{ji}^{(l)}$	Precoding matrix used by Tx_i for transmission to Rx_j in phase l
$\mathbf{C}_{ji}^{(l,k)}$	Precoding matrix used by Tx_i for transmission to Rx_j in part k of phase l
$q^{(l)}$	Number of order-(1,1) symbols generated after phase l
$\mathbf{u}_{i\|j;k}$	Vector of generated order-(1,1) symbols which are available at Tx_i, desired by Rx_j and known at Rx_k
B_i	The i-th decodability bound for the transmitted information symbols
Γ_i	Ratio of the sizes of the signal spaces at Rx_i and $\mathrm{Rx}_{i'}$, $i' = 3 - i$, spanned by the signals intended to Rx_i

The Symmetric MIMO Interference Channel:

$b_i(n)$	Number of information symbols transmitted from Tx_i to Rx_i over communication duration n
$\mathbf{u}_i$	Symbol vector transmitted from Tx_i to Rx_i

$\mathbf{C}_i(t)$	Precoding matrix used by Tx_i for transmission to Rx_i in the t-th channel use
$\mathbf{C}_i^n$	Vertical concatenation of the precoding matrices used by Tx_i for the transmission to Rx_i over communication duration n
$\mathcal{I}_i$	Subspace at Rx_i spanned by the interference
$b_i^{(l)}$	Number of information symbols transmitted from Tx_i to Rx_i in phase l
$b_i^{(l,k)}$	Number of information symbols transmitted from Tx_i to Rx_i in part k of phase l
$b_\Sigma^{(l)}$	Total number of information symbols transmitted in phase l
$q_\Sigma^{(1)}$	Total number of order-2 symbols generated after phase 1
$q_\Sigma^{(2)}$	Total number of order-(2,1) symbols generated after phase 2
$\mathbf{u}_i^{(1)}$	Vector of information symbols transmitted from Tx_i to Rx_i in phase 1
$\mathbf{u}_i^{(1,k)}$	Vector of information symbols transmitted from Tx_i to Rx_i in part k of phase 1
$\mathbf{u}_{i\vert j,k}^{(2)}$	Vector of order-2 symbols transmitted from Tx_i to Rx_j and Rx_k in phase 2
$\mathbf{u}_{i\vert j,k}^{(2,l)}$	Vector of order-2 symbols transmitted from Tx_i to Rx_j and Rx_k in part l of phase 2
$\mathbf{u}_{i\vert j,k;l}^{(3)}$	Vector of order-(2,1) symbols transmitted from Tx_i to Rx_j and Rx_k in phase 3
$\mathbf{u}_{i\vert j,k}$	Vector of generated order-2 symbols which are available at Tx_i and desired by Rx_j and Rx_k
$\mathbf{u}_{i\vert j,k;l}$	Vector of generated order-(2,1) symbols which are available at Tx_i, desired by Rx_j and Rx_k, and known at Rx_l
$\mathbf{C}_i^{(1)}$	Precoding matrix used by Tx_i for transmission to Rx_i in phase 1
$\mathbf{C}_i^{(1,k)}$	Precoding matrix used by Tx_i for transmission to Rx_i in part k of phase 1
$\mathbf{C}_{i\vert j,k}^{(2)}$	Precoding matrix used by Tx_i for transmission to Rx_j and Rx_k in phase 2
$\mathbf{C}_{i\vert j,k}^{(2,l)}$	Precoding matrix used by Tx_i for transmission to Rx_j and Rx_k in part l of phase 2
$\mathbf{C}_{i\vert j,k;l}^{(3)}$	Precoding matrix used by Tx_i for transmission to Rx_j and Rx_k in phase 3
$\delta_i^{(1,k)}$	Size of intersection signal space for signals spanned by Rx_i at two unintended receivers in part k of phase 1

$\mathbf{V}_{j,k;i}^{(1,l)}$	Precoding matrix used by Tx_i, $i=1,2$, which contains coefficients of linear combinations linearly dependent on the linear combinations overheard by Rx_j and Rx_k, $i \neq j \neq k$, in part l of phase 1
$\mathbf{V}_{1,2;3}^{(1)}$	Precoding matrix used by Tx_3, which contains coefficients of linear combinations linearly dependent on the linear combinations overheard by Rx_1 and Rx_2 in part 1 of phase 1
$\mathbf{V}_{ji}^{(1,k)}$	Projection matrix for the signal of Rx_i, $i=1,2$, which was received in part k of phase 1 by Rx_j, $j \neq i$
$\mathbf{V}_{j3}^{(1)}$	Projection matrix for the signal of Rx_3 which was received in part 1 of phase 1 by Rx_j, $j \neq 3$
$B_i^{(l)}$	The i-th decodability bound for the information symbols transmitted in phase l
$\Gamma_{\max}$	Upper bound on the ratio of the sizes of the signal spaces at Rx_i and Rx_j, $i \neq j$, spanned by the signals of two transmitters

The 2-antenna 3-user MISO Broadcast Channel:

$x(t)$	Signal transmitted by Tx
$y_i(t)$	Signal received by Rx_i
$\mathbf{h}_j(t)$	Vector of channel coefficients for Rx_j
$z_j(t)$	Receive noise at Rx_j
I_i	the CSIT for Rx_i
$\lambda_{I_1 I_2 I_3}$	Joint CSIT state probability for the CSIT state $I_1 I_2 I_3$
λ_i	Marginal CSIT state probability for having perfect CSIT for Rx_i
W_i	Message intended to Rx_i
$\hat{W}_i$	Message estimate
$\mathcal{W}_i$	Message set for Rx_i
$R_i(P)$	Achievable rate for Rx_i for given P
$\mathbf{R}(P)$	Achievable rate tuple for given P
$\alpha_i(t)$	Imperfect CSIT quality exponent for the t-channel use
$\phi_i(t)$	Encoding function for the t-th channel use
$\psi_i(t)$	Decoding function for the t-th channel use
P_e	Decoding error probability
$\mathbb{P}(.)$	Probability
$S_i^{d_\Sigma}$	The i-th constituent encoding scheme achieving d_Σ DoF
$u_i^{[k]}$	The k-th symbol transmitted to Rx_i
$\mathbf{u}_i$	The symbol vector transmitted to Rx_i

$\mathbf{u}_i^{[k,l]}$	The symbol vector comprised of the k-th and l-th symbols, transmitted to Rx_i, $k < l$
$u_{i,j}^{[k]}$	The k-th order-2 symbol desired by Rx_i and Rx_j
$\mathbf{u}_{i,j}$	Order-2 symbol vector desired by Rx_i and Rx_j
$u_{i,j;l}^{[k]}$	The k-th order-(2,1) symbol desired by Rx_i and Rx_j and known at Rx_l
$u_{i,j,l}^{[k]}$	The k-th order-3 symbol desired by all receivers
$\mathbf{C}_1(t)$	Precoding matrix for transmission to Rx_1 in the t-th channel use
$\mathbf{c}_i(t)$	Precoding vector for transmission to Rx_i in the t-th channel use
$\mathbf{c}_{23}(t)$	Precoding vector for transmission to Rx_2 and Rx_3, in the t-th channel use
$\mathbf{c}_{i,j}(t)$	Precoding vector for transmission to Rx_i and Rx_j in the t-th channel use
$\mathbf{C}_{i,j}(t)$	Precoding matrix for transmission of order-2 symbols to Rx_i and Rx_j in the t-th channel use
$\mathbf{c}_{i,j;k}(t)$	Precoding vector for transmission of order-(2,1) symbols to Rx_i and Rx_j in the t-th channel use
$\mathbf{c}_{1,2,3}(t)$	Precoding vector for transmission to all receivers in the t-th channel use
γ_i	Precoding scalar for transmission to Rx_i
$\beta_i^{[k]}$	The k-th scaling factor for a symbol received by Rx_i
$L_i^{[k]}(.)$	The k-th linear combination received at Rx_i
$\lambda_{S_i^{d_\Sigma}}$	The fraction of the CS $S_i^{d_\Sigma}$

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Own Publications

1. A. Buzuverov, H. Al-Shatri, and A. Klein, “On the degrees of freedom of the three-user MIMO interference channel with delayed CSIT,” in *Proc. IEEE Information Theory Workshop (ITW)*, Oct. 2015, pp. 239-243.

2. A. Buzuverov, H. Al-Shatri, and A. Klein, “On the achievable degrees of freedom of the MIMO X-channel with delayed CSIT,” in *Proc. IEEE Information Theory Workshop (ITW)*, Sept. 2016, pp. 464-468.

3. A. Buzuverov, H. Al-Shatri, and A. Klein, “Exploiting additional transmit antennas for more degrees of freedom in 3-user MIMO interference channel with delayed CSIT,” in *SCC 2017; 11th International ITG Conference on Systems, Communications and Coding*, Feb. 2017, pp. 1-6.

4. A. Buzuverov and A. Klein, “On the DoF of the 2-antenna 3-user MISO BC with alternating CSIT,” in *SCC 2019; 12th International ITG Conference on Systems, Communications and Coding*, Feb. 2019, pp. 161-166.

Supervised Student Theses

Name	Title of the thesis	Thesis type	Date
Liu, Xiaoze	Downlink Beamforming with Outdated Channel State Information at Transmitters	Master thesis	12/2016

Funding Acknowledgment

The work of A. Buzuverov is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universität Darmstadt.

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Erklärung laut §9 der Promotionsordnung

Ich versichere hiermit, dass ich die vorliegende Dissertation allein und nur unter Verwendung der angegebenen Literatur verfasst habe. Die Arbeit hat bisher noch nicht zu Prüfungszwecken gedient.

Darmstadt, 20. Januar 2019,