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Feasibility Conditions for Relay-Aided Interference Alignment in Partially Connected Networks

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***Abstract*—In this paper, we investigate the feasibility conditions for relay-aided interference alignment in a class of partially connected networks. The considered scenario consists of several single-antenna communicating node pairs and multiple single-antenna amplify-and-forward relays. Some of the direct links between the source and the destination nodes may have zero gain. A two-hop transmission scheme is applied. The relays' scaling factors, along with the transmit and receive filters, are adapted to the channel to null the interference signals at every destination node. However, this would also null the desired useful signal at certain destinations if the number of relays is insufficient. To avoid this, the required number of relays is studied. We show that the required number of relays depends on the rank of the incidence matrix of a graph defined by the topology of the direct links.**

I. Introduction

In recent years, interference alignment (IA) has attracted particular interest. IA aims at achieving the maximum degrees of freedom (DoF) at high signal-to-noise ratio (SNR) in scenarios where the achievable sum-rate is mainly interference-limited. Generally speaking, the following conditions need to be satisfied to accomplish IA:

- **Interference-nulling conditions:** The interference signals shall be aligned in a subspace of the signal space at every destination node.
- **Validity conditions:** The received useful signals shall not fall into the interference subspace and the multiple data streams shall be separable at every destination node.

The IA conditions have been firstly formulated in [1]. An IA solution must satisfy both the interference-nulling conditions and the validity conditions, i.e., be a valid interference-nulling solution.

In [1], it is shown that IA in MIMO interference channels can be accomplished by properly choosing the transmit and receive filters. An iterative algorithm converging to the IA solution is proposed in this paper. For IA in MIMO interference channels, a non-trivial interference-nulling solution fulfils the validity conditions in the almost-sure sense if the channel matrices do not have a particular structure and all channel coefficients are randomly drawn from a continuous distribution [1]. This is because the direct useful channels between the communicating source-destination node pairs do not participate in the interference nulling. The solvability of the interference-nulling conditions can be determined by comparing the number of equations and the number of free variables [2]. This is then proved in [3] for symmetric cases in which the numbers of antennas at all nodes are equal.

A two-hop IA scheme in a three-user interference relay channel utilizing direct links is also introduced in [1]. With the help of the amplify-and-forward relays, one data symbol can be transmitted by each user within two time slots when every node has only a single antenna. In total, $3/2$ DoF can be achieved. For relay-aided IA, the solvability of the interference-nulling conditions is no longer the sole concern of the feasibility. Since the so called relay links are shared by all users and are considered in the interference nulling, the "independency" of the effective useful links cannot be provided if the number of relays is insufficient. In fact, there exist cases in relay-aided IA in which all non-trivial interference-nulling solutions are invalid. The feasibility conditions for relay-aided IA in fully connected networks are studied in [4].

In the present paper, we examine a linear relay-aided IA algorithm with adaptive transmit and receive filters proposed in [5]. Based on this algorithm, we show that if the number of relays is sufficient, valid interference-nulling solutions almost surely exist, i.e., relay-aided IA is almost surely feasible. Furthermore, a randomly picked interference-nulling solution is almost surely a valid one. On the contrary, if the number of relays is insufficient, all interference-nulling solutions are almost surely invalid. The required number of relays to avoid this is studied. We consider a class of partially connected networks where some direct links may be practically zero because of attenuation effects, e.g., path loss and fading. However, every relay is connected to every source and destination node. With the given network topology, a graph theoretic approach is applied to derive the required number of relays.

The rest of this paper is organized as follows. The considered scenario is introduced in Section II. The IA conditions are formulated in Section III. We will then derive the feasibility conditions in Section IV. The graph theoretic approach we used is introduced in Section V, along with a few examples. Finally, we present some simulation results in Section VI and conclude the paper.

II. System Model

Consider an interference channel consisting of K source-destination node pairs and R one-way relays. Every node and every relay is equipped with a single antenna. The amplify-and-forward relaying strategy is used. A two-hop transmission

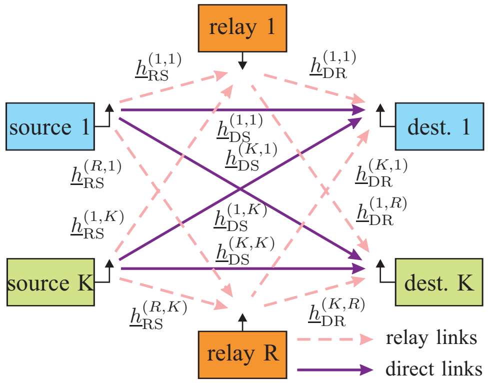

Fig. 1. Two-hop transmission scheme.

scheme as illustrated in Fig. 1 is applied. In the first time slot, every source node transmits a single data symbol to both the relays and the destination nodes. In the second time slot, the source nodes transmit the same data symbols again to the destination nodes while the relays retransmit a scaled version of their received signals to the destination nodes. Furthermore, full channel state information is assumed to be available at the nodes and at the relays. Throughout this paper, $K > 1$ is always assumed. The noise signal is only considered in the simulation results.

Define the channel matrices as $\underline{\mathbf{H}}_{\mathrm{DS}} = [\underline{h}_{\mathrm{DS}}^{(k,j)}]_{K\times K}$, $\underline{\mathbf{H}}_{\mathrm{RS}} = [\underline{h}_{\mathrm{RS}}^{(r,j)}]_{R\times K}$, and $\underline{\mathbf{H}}_{\mathrm{DR}} = [\underline{h}_{\mathrm{DR}}^{(k,r)}]_{K\times R}$, where $\underline{h}_{\mathrm{DS}}^{(k,j)}$, $\underline{h}_{\mathrm{RS}}^{(r,j)}$, and $\underline{h}_{\mathrm{DR}}^{(k,r)}$ denote the channel coefficients from the j-th source node to the k-th destination node, from the j-th source node to the r-th relay and from the r-th relay to the k-th destination node, respectively. The channel coefficients are independently drawn from a continuous distribution over the complex field and are considered to be constant throughout the transmission duration. In the partially connected networks discussed in this paper, some elements of $\underline{\mathbf{H}}_{\mathrm{DS}}$ are set be zero and the other channel coefficients are non-zero with probability 1. Fully connected networks are treated as special cases. Let $\underline{d}_j$ and $\underline{\mathbf{v}}^{(j)} = (\underline{v}_1^{(j)}, \underline{v}_2^{(j)})^{\mathrm{T}}$ denote the transmitted data symbol and the temporal transmit filter of the j-th source node, respectively. Let $\underline{\mathbf{u}}^{(k)} = (\underline{u}_1^{(k)}, \underline{u}_2^{(k)})^{\mathrm{T}}$ and $\hat{\underline{d}}_k$ denote the temporal receive filter and the filter output at the k-th destination node, respectively. Additionally, let the r-th relay's scaling factor be denoted by $\underline{g}^{(r)}$.

Since two time slots are utilized, the virtual channel between the j-th source node and the k-th destination node is a 2×2 MIMO channel and given by

$$\underline{\mathbf{H}}^{(k,j)} = \begin{pmatrix} \underline{h}_{\mathrm{DS}}^{(k,j)} & 0 \\ \sum_{r=1}^{R} \underline{h}_{\mathrm{DR}}^{(k,r)} \underline{g}^{(r)} \underline{h}_{\mathrm{RS}}^{(r,j)} & \underline{h}_{\mathrm{DS}}^{(k,j)} \end{pmatrix}. \quad (1)$$

Therefore, the receive filter output at the k-th destination can be written as

$$\hat{\underline{d}}_k = \underline{\mathbf{u}}^{(k)*\mathrm{T}} \underline{\mathbf{H}}^{(k,k)} \underline{\mathbf{v}}^{(k)} \underline{d}_k + \underline{\mathbf{u}}^{(k)*\mathrm{T}} \sum_{\substack{j=1 \\ j\neq k}}^{K} \underline{\mathbf{H}}^{(k,j)} \underline{\mathbf{v}}^{(j)} \underline{d}_j. \quad (2)$$

In (2), the first term represents the useful signal for the k-th destination node and the second term contains only interferences. Then the IA conditions can be formulated as follows:

$$\underline{\mathbf{u}}^{(k)*\mathrm{T}} \underline{\mathbf{H}}^{(k,j)} \underline{\mathbf{v}}^{(j)} = 0, \quad \forall k, j \in \{1, \ldots, K\}, \quad k \neq j \quad (3)$$
$$\underline{\mathbf{u}}^{(k)*\mathrm{T}} \underline{\mathbf{H}}^{(k,k)} \underline{\mathbf{v}}^{(k)} \neq 0, \quad \forall k \in \{1, \ldots, K\}. \quad (4)$$

III. IA Conditions with Adaptive Filters

A. Interference Nulling

The interference-nulling conditions of (3) are non-linear in the relays' scaling factors and the filter coefficients. But it is natural to assume that $\underline{u}_2^{(k)*} \underline{v}_1^{(j)} \neq 0, \ \forall k, j$. Otherwise, successful transmission would only be possible in very rare scenarios, which do not influence our conclusions. Hence, we exclude this situation from our discussion for simplicity. Under this assumption, (3) can be reformulated as

$$\sum_{r=1}^{R} \underline{h}_{\mathrm{DR}}^{(k,r)} \underline{g}^{(r)} \underline{h}_{\mathrm{RS}}^{(r,j)} + \underline{h}_{\mathrm{DS}}^{(k,j)} \left(\frac{\underline{v}_2^{(j)}}{\underline{v}_1^{(j)}} + \frac{\underline{u}_1^{(k)*}}{\underline{u}_2^{(k)*}} \right) = 0, \ \forall k \neq j. \quad (5)$$

Let $\underline{g}^{(r)}$, $\underline{v}_2^{(j)}/\underline{v}_1^{(j)}$ and $\underline{u}_1^{(k)*}/\underline{u}_2^{(k)*}$ be chosen as the unknown variables. Then the equations of (5) form a linear system of equations in $R + 2K$ variables. Define $\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\underline{\mathbf{H}}_{\mathrm{DL}}$ to be matrices of dimensions $K(K-1) \times R$ and $K(K-1) \times 2K$ as shown in (6) at the top of next page. Using these matrices, the linear system of equations represented by (5) can be rewritten in matrix-vector form as

$$(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}}) \underline{\mathbf{x}} = 0, \quad (7)$$

where $\underline{\mathbf{x}}$ contains the unknown variables. We refer to the solution space of (7) as the interference-nulling solution space denoted by $\mathrm{Null}(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}})$. A simple non-trivial solution of (7) can always be found by setting the relays' scaling factors to zero and choosing the transmit and the receive filters to be pairwise orthogonal. Unfortunately, this solution nulls the useful signals at every destination node and is an invalid solution.

B. Invalid and Valid Solutions

Similarly, rewrite the validity conditions of (4) as

$$\sum_{r=1}^{R} \underline{h}_{\mathrm{DR}}^{(k,r)} \underline{g}^{(r)} \underline{h}_{\mathrm{RS}}^{(r,k)} + \underline{h}_{\mathrm{DS}}^{(k,k)} \left(\frac{\underline{v}_2^{(k)}}{\underline{v}_1^{(k)}} + \frac{\underline{u}_1^{(k)*}}{\underline{u}_2^{(k)*}} \right) \neq 0. \quad (8)$$

Let the interference-nulling solutions which satisfy the inequalities of (8) for all $k = 1, \ldots, K$ be defined as the valid solutions. The sets of the valid and invalid solutions will be given with the help of the following notations.

The expression on the left hand side of (8) is linear in the chosen unknown variables. Let the coefficient vector be denoted by $(\underline{\mathbf{a}}_k | \underline{\mathbf{b}}_k)$, where $\underline{\mathbf{a}}_k$ is a $1 \times R$ row vector with $\underline{h}_{\mathrm{DR}}^{(k,r)} \underline{h}_{\mathrm{RS}}^{(r,k)}$, $r = 1, \ldots, R$ being its elements and $\underline{\mathbf{b}}_k$ is a $1 \times 2K$ row vector with all elements being zero except for the

$$
\underline{\mathbf{H}}_{\mathrm{RL}} = \begin{pmatrix}
\underline{h}_{\mathrm{DR}}^{(2,1)} \underline{h}_{\mathrm{RS}}^{(1,1)} & \cdots & \underline{h}_{\mathrm{DR}}^{(2,R)} \underline{h}_{\mathrm{RS}}^{(R,1)} \\
\vdots & & \vdots \\
\underline{h}_{\mathrm{DR}}^{(K,1)} \underline{h}_{\mathrm{RS}}^{(1,1)} & \cdots & \underline{h}_{\mathrm{DR}}^{(K,R)} \underline{h}_{\mathrm{RS}}^{(R,1)} \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\underline{h}_{\mathrm{DR}}^{(1,1)} \underline{h}_{\mathrm{RS}}^{(1,K)} & \cdots & \underline{h}_{\mathrm{DR}}^{(1,R)} \underline{h}_{\mathrm{RS}}^{(R,K)} \\
\vdots & & \vdots \\
\underline{h}_{\mathrm{DR}}^{(K-1,1)} \underline{h}_{\mathrm{RS}}^{(1,K)} & \cdots & \underline{h}_{\mathrm{DR}}^{(K-1,R)} \underline{h}_{\mathrm{RS}}^{(R,K)}
\end{pmatrix}, \quad
\underline{\mathbf{H}}_{\mathrm{DL}} = \left(\begin{array}{cccc|ccccc}
\underline{h}_{\mathrm{DS}}^{(2,1)} & 0 & \cdots & 0 & 0 & \underline{h}_{\mathrm{DS}}^{(2,1)} & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & & & \ddots & & \ddots \\
\underline{h}_{\mathrm{DS}}^{(K,1)} & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \underline{h}_{\mathrm{DS}}^{(K,1)} \\
 & \ddots & & & \vdots & & & & \vdots \\
 & & \ddots & & \vdots & & & & \vdots \\
0 & \cdots & 0 & \underline{h}_{\mathrm{DS}}^{(1,K)} & \underline{h}_{\mathrm{DS}}^{(1,K)} & 0 & \cdots & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \ddots & & \ddots & \\
0 & \cdots & 0 & \underline{h}_{\mathrm{DS}}^{(K-1,K)} & 0 & \cdots & 0 & \underline{h}_{\mathrm{DS}}^{(K-1,K)} & 0
\end{array}\right) \tag{6}
$$

---

ones in the k-th and the $(K+k)$-th column, which both equal $\underline{h}_{\mathrm{DS}}^{(k,k)}$. Furthermore, define the following extended matrices:

$$
\left(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} \middle| \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)} \right) = \left(\begin{array}{c|c} \underline{\mathbf{H}}_{\mathrm{RL}} & \underline{\mathbf{H}}_{\mathrm{DL}} \\ \hline \underline{\mathbf{a}}_k & \underline{\mathbf{b}}_k \end{array} \right), \quad k = 1, \ldots, K. \tag{9}
$$

Any vector lying in the null space of $(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} | \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)})$ is an invalid solution nulling the useful signal of the k-th user. We refer to $\mathrm{Null}(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} | \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)})$ as the invalid solution subspace with respect to the k-th user. Therefore, the set of the invalid solutions Ψ_{invalid} can be given by

$$
\Psi_{\mathrm{invalid}} = \bigcup_{k=1}^{K} \mathrm{Null} \left(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} \middle| \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)} \right). \tag{10}
$$

The set of the valid solutions Ψ_{valid}, namely the set of IA solutions, is the complement of Ψ_{invalid} in $\mathrm{Null}\left(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}}\right)$.

IV. Feasibility Conditions for Relay-Aided IA

The validity conditions require that each individual vector $(\underline{\mathbf{a}}_k | \underline{\mathbf{b}}_k)$ is linearly independent of the rows in $(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}})$. The following proposition indicates that this is also sufficient in the almost-sure sense.

Proposition 1. *An IA solution exists if and only if*

$$
\mathrm{rank} \left(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} \middle| \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)} \right) = \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}}\right) + 1 \tag{11}
$$

holds for all $k = 1, \ldots, K$. *Then a randomly picked interference-nulling solution of (7) is almost surely a valid one.*

Proof: Since $(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} | \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)})$ is obtained by adding the row $(\underline{\mathbf{a}}_k | \underline{\mathbf{b}}_k)$ to $(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}})$,

$$
\mathrm{Null} \left(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} \middle| \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)} \right) \subseteq \mathrm{Null}\left(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}}\right), \ \forall k \tag{12}
$$

holds. If (11) holds for the k-th user, then the invalid solution subspace with respect to the k-th user represents a hyperplane of Lebesgue measure 0 in the interference-nulling solution space. If (11) holds for all users, then the set of invalid solutions Ψ_{invalid}, which is the union of a finite number of those hyperplanes, is of Lebesgue measure 0 in the interference-nulling solution space as well. Therefore, Proposition 1 follows. If (11) does not hold for at least one user, then the invalid solution subspace with respect to this user is identical to the interference-nulling solution space. Therefore, all interference-nulling solutions are invalid. ■

The following corollary gives a principle for determine whether relays are required at all.

Corollary 1. *For an arbitrary number of relays, a randomly picked interference-nulling solution is almost surely valid if*

$$
\mathrm{rank} \left(\underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)} \right) = \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right) + 1 \tag{13}
$$

holds for all $k = 1, \ldots, K$.

Proof: Clearly, if (13) holds, then (11) holds for arbitrary number of relays. The corollary follows. ■

In order to determine the required number of relays such that (11) is satisfied, $\mathrm{rank}(\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)} | \underline{\widetilde{\mathbf{H}}}_{\mathrm{DL}}^{(k)})$ and $\mathrm{rank}(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}})$ need to be studied. We first argue that in the considered partially connected networks, the matrices $\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)}$ are almost surely of full rank. Note that $\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)}$ are submatrices of $\underline{\mathbf{H}}_{\mathrm{RS}}^{\mathrm{T}} \odot \underline{\mathbf{H}}_{\mathrm{DR}}$, where $\odot$ denotes the Khatri-Rao product [6], which can be understood as the column-wise Kronecker product as well. Corollary 1 in [7] shows that the Khatri-Rao product $\mathbf{A}$ of two matrices whose entries are independently drawn from a continuous distribution is almost surely of full k-rank[1]. This conclusion holds for $\mathbf{A}^{\mathrm{T}}$ as well. This means a submatrix of $\underline{\mathbf{H}}_{\mathrm{RS}}^{\mathrm{T}} \odot \underline{\mathbf{H}}_{\mathrm{DR}}$ consisting of a collection of arbitrarily chosen r rows has rank $\min\{r, R\}$. The matrices $\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)}$ can be obtained by removing K and $K-1$ rows from $\underline{\mathbf{H}}_{\mathrm{RS}}^{\mathrm{T}} \odot \underline{\mathbf{H}}_{\mathrm{DR}}$, respectively. Therefore, $\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\underline{\widetilde{\mathbf{H}}}_{\mathrm{RL}}^{(k)}$, $k = 1, \ldots, K$ are almost surely of full rank.

We then argue that

$$
\mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{RL}} | \underline{\mathbf{H}}_{\mathrm{DL}}\right) = \min\left\{K(K-1), R + \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right)\right\} \tag{14}
$$

[1]Kruskal-rank or k-rank [8]: the k-rank of a matrix $\mathbf{A}$ is r if every r columns of $\mathbf{A}$ are linearly independent and either $\mathbf{A}$ has r columns or $\mathbf{A}$ contains a set of $r+1$ linearly dependent columns.

almost surely holds. $(\underline{\mathbf{H}}_{\mathrm{RL}}|\underline{\mathbf{H}}_{\mathrm{DL}})$ is of dimension $K(K-1) \times (R+2K)$. For $K=2$, it is obvious that $\underline{\mathbf{H}}_{\mathrm{DL}}$ has two independent rows. Equation (14) follows directly. For $K \geq 3$, first consider the case

$$K(K-1) \geq R + \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right). \tag{15}$$

In this case, $\underline{\mathbf{H}}_{\mathrm{RL}}$ is a tall matrix and of rank R. Rewrite $\underline{\mathbf{H}}_{\mathrm{DL}}$ as

$$\underline{\mathbf{H}}_{\mathrm{DL}} = \underline{\mathbf{D}}_{\mathrm{DS}}\mathbf{M}, \tag{16}$$

where $\mathbf{M}$ is obtained by replacing the non-zero entries of $\underline{\mathbf{H}}_{\mathrm{DL}}$ with ones and let $\underline{\mathbf{D}}_{\mathrm{DS}}$ be a diagonal matrix with $(\underline{h}_{\mathrm{DS}}^{(2,1)}, \ldots, \underline{h}_{\mathrm{DS}}^{(K,1)}, \ldots, \underline{h}_{\mathrm{DS}}^{(1,K)}, \ldots, \underline{h}_{\mathrm{DS}}^{(K-1,K)})$ being its diagonal entries. Note that some diagonal entries of $\underline{\mathbf{D}}_{\mathrm{DS}}$ are zeros because of the partial connectivity of the direct links. But replacing them with arbitrary non-zeros cannot hurt the result. When doing so, $\underline{\mathbf{D}}_{\mathrm{DS}}$ becomes invertible. Therefore, $\underline{\mathbf{H}}_{\mathrm{DL}}$ and $\mathbf{M}$ are of the same rank and

$$\mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{RL}}\middle|\,\underline{\mathbf{H}}_{\mathrm{DL}}\right) = \mathrm{rank}\left(\underline{\mathbf{D}}_{\mathrm{DS}}^{-1}\underline{\mathbf{H}}_{\mathrm{RL}}\middle|\,\mathbf{M}\right) \tag{17}$$

holds. Then it suffices to show that the column spaces of $\underline{\mathbf{D}}_{\mathrm{DS}}^{-1}\underline{\mathbf{H}}_{\mathrm{RL}}$ and $\mathbf{M}$ are almost surely disjoint under the condition of (15). One the one hand, the entries of $\mathbf{M}$ are only zeros and ones. In partially connected networks with given topology, $\mathrm{span}\,(\mathbf{M})$ is a fixed subspace of $\mathbb{C}^{K(K-1)}$ and is of dimension $\mathrm{rank}\,(\underline{\mathbf{H}}_{\mathrm{DL}})$. On the other hand, the entries of $\underline{\mathbf{D}}_{\mathrm{DS}}^{-1}\underline{\mathbf{H}}_{\mathrm{RL}}$ depend on the channel coefficients, which are randomly drawn from a continuous distribution. From an engineering point of view, we can conclude that $\mathrm{span}\,(\mathbf{M})$ and $\mathrm{span}\left(\underline{\mathbf{D}}_{\mathrm{DS}}^{-1}\underline{\mathbf{H}}_{\mathrm{RL}}\right)$ are almost surely disjoint as long as the sum of their dimensions does not exceed $K(K-1)$. This result is also supported by numerical simulations. For $K \geq 3$ and $K(K-1) < R + \mathrm{rank}\,(\underline{\mathbf{H}}_{\mathrm{DL}})$, a submatrix of $(\underline{\mathbf{H}}_{\mathrm{RL}}|\underline{\mathbf{H}}_{\mathrm{DL}})$ consisting of $\underline{\mathbf{H}}_{\mathrm{DL}}$ and arbitrary $K(K-1) - \mathrm{rank}\,(\underline{\mathbf{H}}_{\mathrm{DL}})$ columns of $\underline{\mathbf{H}}_{\mathrm{RL}}$ is of rank $K(K-1)$ according to the discussion for the case (15). Hence, (14) almost surely holds. The argument above can be also applied to shown that

$$\mathrm{rank}\left(\widetilde{\underline{\mathbf{H}}}_{\mathrm{RL}}^{(k)}\middle|\,\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}\right) = \min\left\{K(K-1)+1, R + \mathrm{rank}\left(\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}\right)\right\} \tag{18}$$

almost surely holds for all $k = 1, \ldots, K$.

Equipped with these results, we can derive the required number of relays. Define $\mathcal{I}$ to be a subset of the indices of the user pairs for which equation (13) is not satisfied, thus

$$\mathcal{I} = \left\{k \in \{1, \ldots, K\}\,\middle|\,\mathrm{rank}\left(\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}\right) = \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right)\right\}. \tag{19}$$

Regarding $\mathcal{I}$, the feasibility conditions for linear relay-aided IA in the considered partially connected networks can be summarized as follows:

- If $\mathcal{I}$ is the empty set, then (13) holds for all users. Consequently, IA is almost surely feasible with an arbitrary number of relays.
- If $\mathcal{I}$ is non-empty and
$$R \leq K(K-1) - \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right), \tag{20}$$
then by the equations of (14) and (18)
$$\mathrm{rank}\left(\widetilde{\underline{\mathbf{H}}}_{\mathrm{RL}}^{(k)}\middle|\,\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}\right) = \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{RL}}|\underline{\mathbf{H}}_{\mathrm{DL}}\right) \tag{21}$$
almost surely holds for the user pairs indexed by the elements in $\mathcal{I}$. By Proposition 1, IA solutions unlikely exist, i.e., relay-aided IA is almost surely infeasible. Note that for the user pairs which are not indexed by the elements in $\mathcal{I}$, the useful signals would unlikely be nulled and $1/2$ DoF per user are almost always achievable.
- If $\mathcal{I}$ is non-empty and
$$R \geq K(K-1) - \mathrm{rank}\left(\underline{\mathbf{H}}_{\mathrm{DL}}\right) + 1, \tag{22}$$
then (11) almost surely holds for all users. IA is almost surely feasible and the total achievable DoF are $K/2$.

V. Determining the Required Number of Relays

In this section, we propose a graph theoretic approach to determine the rank of $\underline{\mathbf{H}}_{\mathrm{DL}}$ and $\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}$ as well as the required number of relays.

Define a bipartite undirected graph $G[S, D, E]$ whose vertex subsets $S = \{s_1, \ldots, s_K\}$ and $D = \{d_1, \ldots, d_K\}$ represent the source and the destination nodes, respectively. The non-zero direct interference links form the set of edges E. For instance, the diagram of G in a three-user partially connected network is illustrated in Fig. 2a. The edge-vertex incidence matrix $\mathbf{M}_G$ of G can be obtained by removing the zero rows from the matrix $\mathbf{M}$ in (16). Hence, $\underline{\mathbf{H}}_{\mathrm{DL}}$ and $\mathbf{M}_G$ are of the same rank.

Based on G, define the graphs $G_k[S, D, E_k]$, $k = 1, \ldots, K$ which are obtained by adding one edge indicating the direct useful link between the k-th user pair to G, respectively. Additionally, G_k is identical to G if $\underline{h}_{\mathrm{DS}}^{(k,k)}$ equals zero. Consider the example shown in Fig. 2a. We further assume that the direct useful link between s_3 and d_3 is zero. Then the diagrams of G_1, G_2 and G_3 of the partially connected network can be illustrated in Fig. 2b, 2c and 2d, respectively. Let the incidence matrix of G_k be denoted by $\mathbf{M}_{G_k}$. Then $\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}$ and $\mathbf{M}_{G_k}$ have the same rank.

To determine the rank of the incidence matrix of a graph, one can count the number of edges of a maximal forest in the graph, i.e., a maximal acyclic subgraph including all vertices [9]. In the following, we examine the feasibility of relay-aided IA in three scenarios using the proposed method.

Scenario 1. Consider a K-user fully connected network where $K \geq 3$. The corresponding graphs G and G_k of such a network are connected graphs with $2K$ vertices. Any spanning tree of each of those graphs has $2K-1$ edges [9]. Therefore, all matrices $\widetilde{\underline{\mathbf{H}}}_{\mathrm{DL}}^{(k)}$ and $\underline{\mathbf{H}}_{\mathrm{DL}}$ are of rank $2K-1$. According to (22), the required number of relays is given by

$$R \geq K^2 - 3K + 2. \tag{23}$$

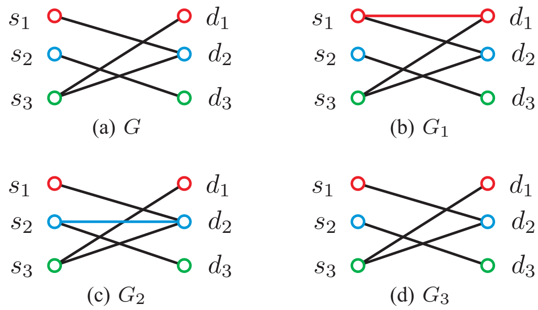

Fig. 2. Diagrams of G, G_1, G_2 and G_3 in a three-user partially connected network.

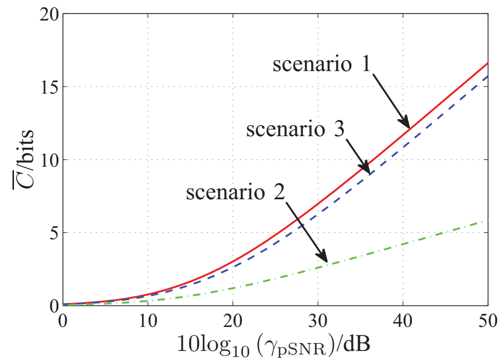

Fig. 3. The sum-rate as a function of the pSNR.

Scenario 2. Consider a three-user partially connected network whose topology of the direct links is shown in Fig. 2. Two relays are connected to every source and destination node. A maximal forest in G, G_1, G_2 and G_3 has 4, 4, 5 and 4 edges, respectively. The invalid solution subspaces with respect to the 1st and the 3rd user are almost surely identical to the interference-nulling solution space. Hence, relay-aided IA is almost surely infeasible. However, the useful signal of the 2nd user would unlikely be nulled by a randomly picked interference-nulling solution with arbitrary number of relays.

Scenario 3. Consider the three-user partially connected network whose topology of the direct links is the same as the one in scenario 2. Three relays are connected to every source and destination node. Then the inequality of (22) is satisfied and relay-aide IA is almost surely feasible.

VI. Numerical Results

In this section, the average sum-rate per time-slot is taken as a measure of the performance of the considered relay-aided IA scheme, thus

$$C = \frac{1}{2}\sum_{k=1}^{K} \mathrm{ld}\left(1+\gamma_k\right), \tag{24}$$

where γ_k is the SNR at the k-th destination node. Define the pseudo signal-to-noise ratio (γ_{pSNR}) to be the ratio of the total energy transmitted in two time-slots by the source nodes and the relays to the noise variance at a destination node [5]. Furthermore, the non-zero channel coefficients are assumed to be i.i.d. Rayleigh fading with unit average channel gain. Additive white Gaussian noise is assumed at both the relays and the destination nodes. The system performances in the three scenarios introduced in Section V are studied. In scenario 1, three user pairs and two relays are assumed. The average performance achieved by randomly chosen interference-nulling solutions over a large number of channel realizations is illustrated in Fig. 3. In the three-user fully connected network with 2 relays, the chosen interference-nulling solution is almost surely valid. The total achieved DoF per time-slot is $3/2$. In scenario 2, only the useful signal of the 2nd user is unlikely nulled by the chosen solution. Therefore, the achieved DoF is $1/2$. However, three relays are sufficient for scenario 3. Consequently, $3/2$ DoF can be achieved.

VII. Conclusion

We analyze the feasibility conditions for relay-aided IA with adaptive transmit and receive filters in partially connected networks. We show that the validity conditions are essential for relay-aided IA. If the number of relays is sufficient, a randomly picked interference-nulling solution is almost surely valid. The required number of relays can be determined using graph theory. Note that in partially connected networks more relays may be required as compared to the fully connected networks.

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