Optimum MMSE Detection with Correlated Random Noise Variance in OFDM Systems

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***Abstract*— In most previous research work on data detection in mobile radio systems the noise is assumed to be Gaussian distributed with known noise variance, and a separate variance estimation is required in realistic scenarios. In order to avoid the additional variance estimator and to obtain better system performance, a novel optimum minimum mean square error (MMSE) detector exploiting noise statistics, i.e., random noise variances which are multivariate Chi-square distributed with exponential correlation in different subcarriers of an OFDM system, is proposed.**

I. Introduction

It is well known that a better quality of service (QoS) is required in 4G mobile radio systems. Improving data detection is always a promising issue in order to obtain better system performance and to achieve the required QoS in mobile radio systems. In most research work on mobile radio systems, data detection assuming Gaussian distributed noise with known noise variance is applied [1], [2]. However, the noise variance is changing as a random variance in many realistic communication systems, and consequently the noise is not strictly Gaussian distributed anymore. In fact, the exact value of the noise variance has to be estimated separately at each time slot, and only when the estimation is based on a sufficient number of signals, good performance of data detection can be ensured [3]. As we know, in most cases the data symbol alphabet is known by the data detector and the approximate distribution of the noise variance can be easily obtained in realistic scenarios. Data detection based on non-Gaussian noise has received a lot of research interest in the recent years [4], and in this paper we will discuss the data detection based on noise statistics with known probability density function (PDF) of the noise variance. The model of the noise statistics is derived following the ideas of the inverse Gaussian transform of the PDF of the noise variance [5], [6]. Exploiting the data symbol alphabet and the limited knowledge of the noise statistics, an optimum MMSE detector is proposed in this paper. Concepts similar to our optimum MMSE approach have been proposed in some other application fields under the names Bayesian optimum radar detection in [4], nonlinear MMSE estimates in [7], and asymptotically optimal nonlinear MMSE multiuser detection in [8]. In order to explain the principle of our proposal for data detection in a simple way, in this paper we consider the application of the optimum MMSE detection to the data transmission in an OFDM system with several signals transmitted simultaneously in different subcarriers [9]. Exploiting the effect of correlated fading, we assume that the noises in individual subcarriers have correlated random variances described by a multivariate Chi-square distribution with exponential correlation. The derivation of the PDF of the noise variance considering an exponential correlation matrix is based on previous work in [10] and [11]. The promising approach of optimum MMSE detection exploiting the realistic noise model with correlated random noise variances is considered as the main contribution of this paper. The application of our proposal is not limited to data detection in OFDM systems, but generally also lies in other fields of communication systems. Especially, it is expected that our proposal can contribute to further research about data estimate refinement for iterative soft interference cancelation in cellular systems [12], decision feedback receivers [13] and channel estimation [7] for future communication systems.

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II. System model

In order to explain our proposal, i.e., optimum MMSE detection with correlated random noise variance, in a simple way, data transmission in a real-valued OFDM system is considered in this paper. This is not a strong restriction as any complex-valued system can equivalently be modeled as a real-valued system with double dimension.

Applying BPSK modulation in the OFDM transmitter, the transmitted vector in the frequency domain is equal to the data vector

$$\mathbf{d} = (d_1 \dots d_K)^{\mathrm{T}},\ d_k \in \{-1; +1\},\ k = 1 \dots K\ . \quad (1)$$

At every OFDM time slot, the data symbols d_k included in the data vector $\mathbf{d}$ of dimension K are transmitted over K subcarriers of the OFDM system with individual channel coefficients h_k. Assuming that there are no inter-subcarrier interferences, only noise signals n_k are considered as disturbances in the system model. It is assumed that the noise signals n_k are independently Gaussian distributed if the values of the noise variance σ_k^2 are known. The conditional PDFs of the noise signals n_k are

$$\mathrm{p}(n_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi\sigma_k^2}} e^{\frac{-{n_k}^2}{2\sigma_k^2}}, k = 1 \ldots K, \tag{2}$$

where different noise variances σ_k^2 are assumed in each subcarrier. We can easily obtain the conditional joint PDF of the noise vector from (2) as

$$\mathrm{p}(\mathbf{n}|\sigma_1^2, \ldots, \sigma_K^2) = \frac{e^{-\frac{1}{2}\left(n_1^2/\sigma_1^2 + \cdots + n_K^2/\sigma_K^2\right)}}{(\sqrt{2\pi})^K \cdot \sigma_1 \cdots \sigma_K} . \tag{3}$$

At the receiver side with the additive noise signals n_k in every subcarrier k, we obtain the received signals e_k which are included in the received vector $\mathbf{e}$. With the data vector $\mathbf{d}$, the noise vector $\mathbf{n}$, and the channel matrix

$$\mathbf{H} = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_K \end{pmatrix} \tag{4}$$

of the OFDM system, the real valued system model reads

$$\mathbf{e} = \mathbf{H} \cdot \mathbf{d} + \mathbf{n} . \tag{5}$$

III. Noise model with exponentially correlated multivariate Chi-square distributed variance

In this section, we will focus on the model of the noise statistics based on the above system model. As mentioned before, the noise signals are assumed to be conditional Gaussian distributed in the case that the noise variance in each subcarrier is a priori known. However, in many realistic scenarios of interest, the noise variance is not a priori known but should be considered as a random variable. In this paper, we assume that there are R noise sources having the same given power. The received noise in each subcarrier results from the superposition of the R noise signals after propagating through R individual independently fading channels. The noise from a certain noise source is relevant for different subcarriers with correlated fading channels. In the case that the individual real-valued channel coefficients are identically Gaussian distributed with mean 0 and variance Ω, the total channel energy is correlated multivariate Chi-square distributed with R degrees of freedom. Consequently, the power of the received noise, i.e., the noise variance, after passing through these channels is also correlated multivariate Chi-square distributed with R degrees of freedom if the noise sources have identical power. That is to say, the values of the noise variance in different subcarriers are in general statistically correlated with each other as a result of the channel correlations in wireless communication systems. Based on previous work on correlated fading channels [10], it follows that the statistics of the correlated noise variance can be described by a multivariate Chi-square distribution with exponential correlation. In the scenario of multi-channel reception of the noise signals, the correlation between pairs of noise variances in two subcarriers decays as the frequency separation of the subcarriers increases. According to this fact, the exponential correlation coefficient of the noise variances σ_i^2 and σ_j^2 in the two subcarriers of this model can be described by

$$\rho_{i,j} = \frac{\mathrm{cov}(\sigma_i^2, \sigma_j^2)}{\sqrt{\mathrm{var}(\sigma_i^2)\mathrm{var}(\sigma_j^2)}} = \rho^{|i-j|}, \quad 0 < \rho < 1 . \tag{6}$$

Under the above assumption, we can derive the joint PDF of the exponentially correlated multivariate Chi-square distributed noise variances using [10] as

$$\mathrm{p}\left(\sigma_1^2, \ldots, \sigma_K^2\right) = \frac{\sigma_1^{(\frac{R}{2}-1)} \sigma_K^{\frac{R}{2}} e^{-\frac{\sigma_1^2 + \sigma_K^2 + (1+\rho^2)\sum_{k=2}^{K-1}\sigma_k^2}{2\Omega(1-\rho^2)}}}{(2\Omega)^{(K+\frac{R}{2}-1)} \Gamma(\frac{R}{2}) (1-\rho^2)^{\frac{R(K-1)}{2}}}$$

$$\cdot \prod_{k=1}^{K-1} \left(\frac{\rho}{1-\rho^2}\right)^{1-\frac{R}{2}} I_{(\frac{R}{2}-1)}\left(\frac{\rho}{\Omega(1-\rho^2)} \sigma_k \sigma_{k+1}\right) , \tag{7}$$

where $I_\alpha(x)$ indicates the modified Bessel function of the first kind and order α. The PDF of the noise vector $\mathbf{n}$ is calculated as the marginal PDF from the conditional joint PDF of the noise vector $\mathrm{p}(\mathbf{n}|\sigma_1^2, \ldots, \sigma_K^2)$ in (3) using (7) as

$$\mathrm{p}(\mathbf{n}) = \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{K} \mathrm{p}(\mathbf{n}|\sigma_1^2, \ldots, \sigma_K^2) \cdot \mathrm{p}\left(\sigma_1^2, \ldots, \sigma_K^2\right) \mathrm{d}\sigma_1^2 \cdots \mathrm{d}\sigma_K^2 . \tag{8}$$

With

$$\varphi(a, b, c) = 2c^{\frac{a+1}{2}} b^{-\frac{a+1}{2}} K_{(a+1)}(2\sqrt{bc}), \tag{9}$$

where $K_\alpha(x)$ indicates the modified Bessel function of the second kind and order α, we obtain

$$\mathrm{p}(\mathbf{n}) = \frac{(1-\rho^2)^{\frac{R(1-K)}{2}}}{(\sqrt{2\pi})^K (2\Omega)^{\frac{RK}{2}} \Gamma(\frac{R}{2})} \cdot \sum_{i_1, \ldots, i_{K-1}=0}^{\infty} \left(\prod_{j=1}^{K-1} \frac{1}{i_j!} \right.$$

$$\left. \cdot \frac{\left(\frac{\rho}{2\Omega(1-\rho^2)}\right)^{2i_j}}{\Gamma(i_j + \frac{R}{2})} \cdot \varphi\left(\frac{R-3}{2} + i_1, \frac{1}{2\Omega(1-\rho^2)}, \frac{n_1^2}{2}\right) \cdots$$

$$\cdot\varphi\left(\frac{R-3}{2}+i_{k-2}+i_{k-1},\frac{1+\rho^2}{2\Omega(1-\rho^2)},\frac{n_{k-1}^2}{2}\right)\cdots$$

$$\cdot\varphi\left(\frac{R-2}{2}+i_{K-1},\frac{1}{2\Omega(1-\rho^2)},\frac{n_K^2}{2}\right)\Bigg) \qquad (10)$$

in the case of $K > 2$, and

$$\mathrm{p}(\mathbf{n}) = \frac{(1-\rho^2)^{-\frac{R}{2}}}{2\pi(2\Omega)^R\Gamma(\frac{R}{2})}\cdot\sum_{i=0}^{\infty}\frac{\left(\frac{\rho}{2\Omega(1-\rho^2)}\right)^{2i}}{i!\Gamma(i+\frac{R}{2})}$$
$$\cdot\varphi\left(\frac{R-3}{2}+i,\frac{1}{2\Omega(1-\rho^2)},\frac{n_1^2}{2}\right)$$
$$\cdot\varphi\left(\frac{R-2}{2}+i,\frac{1}{2\Omega(1-\rho^2)},\frac{n_2^2}{2}\right) \qquad (11)$$

in the case of $K = 2$. The above model of the noise statistics with appropriate parameters derived from realistic scenarios is very promising in data detection with non-Gaussian noise. In the following section, this model is applied to the optimum MMSE detector for an OFDM system.

IV. Optimum MMSE data detection

In the OFDM system model introduced in Section II, the received signal e_k in subcarrier k is obtained from (5) as

$$e_k = \underbrace{h_k \cdot d_k}_{y_k} + n_k \ , \qquad (12)$$

where we introduce y_k as the received useful signal. As channel estimation is not considered in this paper, we assume the channel coefficient h_k in each subcarrier to be perfectly known by the receiver. On one side, in each subcarrier the value of the received useful signal y_k can be directly obtained from the corresponding value d_k of the data symbol considering the scaling factor h_k. Applying BPSK modulation for the data transmission in the real-valued OFDM system, we obtain the symbol alphabet

$$\mathbb{V}_{y_k} = \{v_1, v_2\} = \{-h_k, h_k\} \subset \mathbb{R} \qquad (13)$$

of the received useful signal y_k at subcarrier k with the channel coefficient h_k. On the other side, from the estimate of the received useful signal $\hat{y}_k$, we can directly obtain the data estimate

$$\hat{d}_k = \hat{y}_k / h_k \ , \qquad (14)$$

considering the scaling factor h_k. Data detection of the data symbols d_k in such a system is equivalent to data detection of the received useful signals from the received signals e_k including the useful signals and the noise signals.

Aiming at minimizing the mean square error, the estimate $\hat{d}_k$ of the data symbol d_k reads

$$\hat{d}_k = \underset{\hat{d}_k}{\mathrm{argmin}}\left\{\mathrm{E}\left\{(\hat{d}_k - d_k)^2|\mathbf{e}\right\}\right\} \ . \qquad (15)$$

Considering the scaling factor, i.e., channel coefficient h_k, we rewrite (15) as

$$\hat{d}_k = \underset{\hat{d}_k}{\mathrm{argmin}}\left\{\mathrm{E}\left\{(\underbrace{h_k\cdot\hat{d}_k}_{\hat{y}_k} - \underbrace{h_k\cdot d_k}_{y_k})^2|\mathbf{e}\right\}\right\} \ . \qquad (16)$$

Obviously, the MMSE detection of the data symbol in (16) is equivalent to the MMSE detection of the received useful signal in

$$\hat{y}_k = \underset{\hat{y}_k}{\mathrm{argmin}}\left\{\mathrm{E}\left\{(\hat{y}_k - y_k)^2|\mathbf{e}\right\}\right\} \ . \qquad (17)$$

In order to minimize

$$\mathrm{E}\left\{(\hat{y}_k - y_k)^2\middle|\mathbf{e}\right\} = (\hat{y}_k - \mathrm{E}\{y_k|\mathbf{e}\})^2 + \mathrm{var}\left\{y_k|\mathbf{e}\right\} \ , \qquad (18)$$

we have to choose

$$\hat{y}_k = \mathrm{E}\{y_k|\mathbf{e}\} \ . \qquad (19)$$

It should be mentioned that this MMSE detector also maximizes the output signal-to-noise ratio

$$\gamma_{\mathrm{out},k} = \frac{(d_k)^2}{\mathrm{E}\{(\hat{d}_k - d_k)^2\}} = \frac{(y_k)^2}{\mathrm{E}\left\{(\hat{y}_k - y_k)^2\right\}} \ . \qquad (20)$$

According to (19), the estimate of received useful signal is obtained by exploiting the knowledge of the symbol alphabet and the received vector $\mathbf{e}$ including the useful signals and the noise signals. With respect to different levels of knowledge of the noise statistics, the MMSE detector is implemented in various ways as shown in the following.

A. Optimum MMSE detector with known noise variance

The rationale of the optimum MMSE detector is to maximize $\gamma_{\mathrm{out},k}$ in (20). In the best case that the value of the noise variance σ_k^2 in each subcarrier k is perfectly known by the MMSE detector, the received signals $e_k, k = 1 \ldots K$, in different subcarriers are independently Gaussian distributed according to the system model introduced in Section II. Therefore, $\hat{y}_k$ depends only on the received signal in the same subcarrier, and (19) can be simplified as

$$\hat{y}_k = \mathrm{E}\{y_k|e_k\} \ . \qquad (21)$$

Assuming the symbols in the received useful signal symbol alphabet $\mathbb{V}_{y_k}$ in (13) have equal a priori probabilities $\mathrm{P}(y_k = v_1) = \mathrm{P}(y_k = v_2) = \frac{1}{2}$, (21) can be calculated as

$$\hat{y}_k = \frac{\sum_{m=1}^{2} v_m\, \mathrm{p}(e_k|y_k = v_m)\,\mathrm{P}(v_m)}{\sum_{m=1}^{2}\mathrm{p}(e_k|y_k = v_m)\,\mathrm{P}(v_m)} = \frac{\sum_{m=1}^{2} v_m\, e^{-\frac{(e_k - v_m)^2}{2\sigma_k^2}}}{\sum_{m=1}^{2} e^{-\frac{(e_k - v_m)^2}{2\sigma_k^2}}} \ . \qquad (22)$$

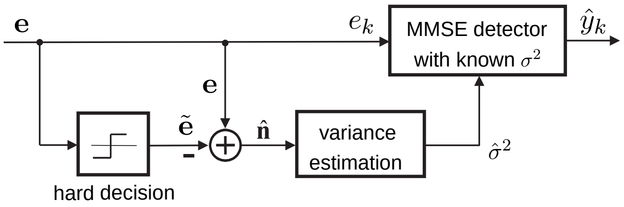

Fig. 1. soft quantizer with separate noise variance estimation

B. MMSE detection with separate noise variance estimation

In most realistic communication systems, the value of the noise variance is not a priori known by the receiver. Many state of the art data detectors have a separate noise variance estimator. For simplicity, we assume all subcarriers have the same noise variance σ^2, and consequently a MMSE detector based on the optimum MMSE detector introduced in Section IV-A with a separate noise variance estimator can be implemented as shown in Fig. 1. The initial estimated data vector is obtained through a hard decision as

$$\tilde{\mathbf{e}} = (\tilde{e}_1 \ldots \tilde{e}_K)^{\mathrm{T}},\ \tilde{e}_k = h_k \cdot \mathrm{sign}(e_k),\ k = 1 \ldots K. \quad (23)$$

The variance estimator exploits the resulting estimated noise vector in K subcarriers to estimate the variance as

$$\hat{\sigma}^2 = (\hat{\mathbf{n}}^{\mathrm{T}} \cdot \hat{\mathbf{n}})/K = ((\mathbf{e} - \tilde{\mathbf{e}})^{\mathrm{T}} \cdot (\mathbf{e} - \tilde{\mathbf{e}}))/K \ . \quad (24)$$

Taking the estimated noise variance, the MMSE detector works in the same way as introduced in Section IV-A.

C. Optimum MMSE detector with known PDF of the exponentially correlated noise variance

Only if a sufficient number of signals are considered in the variance estimation, good performance can be expected in the MMSE detector shown in Section IV-B. If the noise variances in different subcarriers don't have the same value, the MMSE detector in Section IV-B can not be applied. One promising approach for the MMSE detector to obtain better system performance and to avoid the separate variance estimation is to make full use of the limited knowledge of the noise statistics, i.e., the PDF of the noise variance. Applying the realistic noise statistic model obtained in Section III exploiting the known PDF, i.e., the multivariate Chi-square distribution with exponential correlation, of the noise variance, an optimum MMSE detector minimizing the mean square error and maximizing the output signal-to-noise ratio $\gamma_{\mathrm{out},k}$ can be obtained. From the rationale in (19), exploiting the received useful data vector alphabet

$$\mathbb{U}_{\mathbf{y}} = \{\mathbf{u}_1 \ldots \mathbf{u}_M\} \quad (25)$$

of cardinality $M = 2^k$ with

$$\mathbf{u}_m = (u_{m,1} \ldots u_{m,K})^{\mathrm{T}},\ u_{m,k} \in \mathbb{V}_{y_k},\ k = 1 \ldots K, \quad (26)$$

we obtain the received useful data estimates

$$\hat{y}_k = \frac{\sum_{m=1}^{M=2^K} u_{m,k}\, \mathrm{p}(\mathbf{e}|\mathbf{y} = \mathbf{u}_m)\, \mathrm{P}(\mathbf{u}_m)}{\sum_{m=1}^{M=2^K} \mathrm{p}(\mathbf{e}|\mathbf{y} = \mathbf{u}_m)\, \mathrm{P}(\mathbf{u}_m)} \ , \quad (27)$$

where

$$\mathrm{p}(\mathbf{e}|\mathbf{y} = \mathbf{u}_m) = \mathrm{p}(\mathbf{e} - \mathbf{u}_m) = \mathrm{p}(\mathbf{n}) \quad (28)$$

can be obtained from (10) or (11). Assuming equal a priori probabilities $\mathrm{P}(\mathbf{u}_m)$, (27) can be rewritten as

$$\hat{y}_k = \frac{\sum_{m=1}^{M=2^K} u_{m,k}\, \mathrm{p}(\mathbf{e} - \mathbf{u}_m)}{\sum_{m=1}^{M=2^K} \mathrm{p}(\mathbf{e} - \mathbf{u}_m)} \ . \quad (29)$$

V. Numerical results

The performance of our proposal is assessed with the help of some numerical results in this section. For first investigations, data transmission applying BPSK modulation with unit data symbol energy in two subcarriers of the OFDM system with noise coming from two noise sources is considered, i.e., $K = 2$ and $R = 2$ is assumed in (11) for the PDF of the noise vector. In order to obtain numerical results, we make a truncation of the infinite series in (11) to the first I terms. It can be verified that the error E_I resulting from the truncation can meet any desired accuracy when I is large enough. An upper bound for the error is obtained as

$$E_I < A \cdot \sum_{i=I}^{\infty} (\rho^2)^i = B \cdot \rho^{2I},\ \ 0 < \rho < 1 \ , \quad (30)$$

where A and B are two constant factors independent from i. The PDF of the noise vector is shown in Fig. 2 considering the first $I = 60$ terms of the infinite series in (11). Exploiting the obtained PDF of the noise and the data alphabet of the useful signal based on the proposed optimum MMSE detection, the estimate function $\hat{y}_1$ of the received useful signals described by (29) is shown in Fig. 3. Since the noise signals in different subcarriers are correlated with each other as a result of their exponentially correlated noise variances, the estimate $\hat{y}_1$ of useful signal in subcarrier 1 depends not only on the received signal e_1 in subcarrier 1, but also on the received signal e_2 in subcarrier 2.

As one special case of the noise with correlated variance, we assume that all the subcarriers have the same noise variance, i.e., $\sigma_k^2 = \sigma^2$. The numerical results for BPSK modulation with respect to the CDFs